Crystallized structure for level 0 part of modified quantum affine algebra

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Abstract

Crystal base of the level 0 part of the modified quantum affine algebra $\widetilde{U}_q(\widehat{sl_2})_0$ is given by path. Weyl group actions, extremal vectors and crystal structure of all irreducible components are described explicitly.

1 Introduction

The modified quantum algebra, which is denoted $\widetilde{U}_q(\mathfrak{g})$, was introduced in [1] for GL_n-case and in [7] for general case. In [14], G.Lusztig showed the existence of canonical (crystal) base of modified quantum algebras for general Lie algebra.

In [9], M.Kashiwara described detailed crystal structure of the modified quantum algebras, in particular, he gave the Peter-Weyl type decomposition theorem for the cases that $\mathfrak{g}$ is finite type and affine type with non-zero level(=central charge) parts. But, in [9], [10], it is mentioned that the structure of level 0 part for affine type is still unclear. By the definition of the modified quantum algebra (3.1) and (3.2), we know that originally $\widetilde{U}_q(\mathfrak{g})$ is neither a highest nor a lowest weight module. Nevertheless, if $\mathfrak{g}$ is affine, we can apply the powerful tool : theory of integrable highest (resp. lowest) weight modules to the positive (resp. negative) level part $\widetilde{U}_q(\mathfrak{g})_+ := \oplus_{\langle c,\lambda\rangle>0} U_q(\mathfrak{g})a_\lambda$ (resp. $\widetilde{U}_q(\mathfrak{g})_- := \oplus_{\langle c,\lambda\rangle<0} U_q(\mathfrak{g})a_\lambda$) by virtue of Weyl group actions on crystal bases, where c is a canonical central element of $\mathfrak{g}$. But, in the level 0 case, there is no such a tool. However, even in level 0 case, it is still a good way to consider Weyl group actions on crystal bases. Classification of 'extremal vectors' (Definition 3.6) is a crucial point in this paper. By applying this classification to "path" realization, we can clarify crystallized structure of the level 0 part of the modified quantum algbra for $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ case and give an explicit description of its

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every connected component as a crystal graph. The Peter-Weyl type theorem for this case will be given in the forthcoming paper.

The path realization for the level 0 part of the modified quantum algebra for $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$ case has an another feature, which is a physical one. A set of "path" is like the following thing:

$$\left\{ (\cdots, i_k, i_{k+1}, \cdots,) \, ; \begin{array}{l} i_k \in \mathbf{Z},\, i_k = 0\,(k \ll 0), \\ i_k = -i_{k+1}\,(k \gg 0). \end{array} \right\}. \tag{1.1}$$

Meanwhile, there is so called 'XXZ type chain model', which is a kind of physical model on the following space:

$$\mathfrak{F} = (\cdots \otimes \mathbf{C}^{l+1} \otimes \mathbf{C}^{l+1} \otimes \mathbf{C}^{l+1} \otimes \cdots)^*,$$

where $\mathbf{C}^{l+1}$ has a basis $\{(i)\}_{i=0,\cdots,l}$ and the notation $(\cdots)^*$ implies the condition that $\mathfrak{F}$ is spanned by vectors $\cdots \otimes (i_k) \otimes (i_{k+1}) \otimes \cdots$ with $i_k + i_{k+1} = l$ for $|k| \gg 0$. We can see that this condition is similar to the condition in (1.1). It is known that the space $\mathfrak{F}$ has a $U_q(\widehat{\mathfrak{sl}_2})$-module structure. In fact, in [2] and [3], this space is realized as

$$\mathfrak{F} = \bigoplus_{\langle c,\zeta\rangle = \langle c,\mu\rangle = l} V(\lambda) \otimes V(-\mu),$$

where $V(\zeta)$ (resp. $V(-\mu)$) is an integrable highest (resp. lowest) weight module. By considering the map $\pi_{\zeta,\mu}$ in (3.3), we can deduce that $U_q(\mathfrak{g})a_\lambda$ is a kind of limit of $\mathfrak{F}$ and Theorem 3.1 guarantees that such a deduction is valid in the crystallized space. For such a limit, in [15] we gave some related algebra structure and its representation theory. But in this paper, we do not touch this subject any more.

Let us see the organization of this paper. In Sec.2, we shall give the definitions of quantized enveloping algebra and crystal. In Sec.3, we introduce modified quantum algebra, its crystal base and Wely group actions. From Sec.4 to Sec.7, we consider the specific case $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$. In Sec.4, we study affinization of classical crystal and give a classification of extremal vectors in $B^{\otimes n}$, where $B = \{\pm\}$ is the two-dimensional crystal, called 'spin'. In Sec.5, we shall give "path realization" of $U_q(\mathfrak{g})a_\lambda$ with level of $\lambda = 0$ and introduce notions of 'domain' and 'wall', which play a crucial role in this paper. We also describe the actions of $\tilde{e}_i$ and $\tilde{f}_i$ on a path. In Sec.6, we give the path-spin correspondence, which is a morphism of classical crystal between paths and spins. In Sec.7, first of all, we shall introduce some parametrizations which are necessary to describe connected components in $B(\widetilde{U}_q(\mathfrak{g}))$. Then we shall give explicit crystallized structure of $\widetilde{U}_q(\mathfrak{g})$ by classifying all extremal vectors in $\widetilde{U}_q(\mathfrak{g})$.

The author would like to acknowledge professor Masaki Kashiwara for his helpful advises. This work was partly done during the stay of the author at Northeastern University. He is grateful to professor Andrei Zelevinsky for his kind hospitality.

2 Notations and Preliminaries

2.1 Definition of $U_q(\mathfrak{g})$

We shall recall the definition of the quantized universal enveloping algebra. First, let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra over $\mathbf{Q}$ with a Cartan subalgebra $\mathfrak{t}$, $\{\alpha_i \in \mathfrak{t}^*\}_{i\in I}$ the set of simple roots and $\{h_i \in \mathfrak{t}\}_{i\in I}$ the set of coroots, where I is a finite index set. We define an inner product on $\mathfrak{t}^*$ such that $(\alpha_i, \alpha_i) \in \mathbf{Z}_{\geq 0}$ and $\langle h_i, \lambda\rangle = 2(\alpha_i, \lambda)/(\alpha_i, \alpha_i)$ for $\lambda \in \mathfrak{t}^*$. Set $Q = \oplus_i \mathbf{Z}\alpha_i$, $Q_+ = \oplus_i \mathbf{Z}_{\geq 0}\alpha_i$ and $Q_- = -Q_+$. We call Q a root lattice. Let P a lattice of $\mathfrak{t}^*$ *i.e.* a free $\mathbf{Z}$-submodule of $\mathfrak{t}^*$ such that $\mathfrak{t}^* \cong \mathbf{Q} \otimes_{\mathbf{Z}} P$, and $P^* = \{h \in \mathfrak{t} | \langle h, P\rangle \subset \mathbf{Z}\}$. We set $P_+ = \{\lambda \in P | \langle \lambda, h_i\rangle \geq 0 \text{ for any } i \in I\}$. An element of P(resp.P_+) is called a integral weight (resp. dominant integral weight).

The quantized enveloping algebra $U_q(\mathfrak{g})$ is an associative $\mathbf{Q}(q)$-algebra generated by e_i, $f_i (i \in I)$ and $q^h (h \in P^*)$ satisfying the following relations:

$$q^0 = 1, \quad \text{and} \quad q^h q^{h'} = q^{h+h'}, \tag{2.1}$$

$$q^h e_i q^{-h} = q^{\langle h,\alpha_i\rangle} e_i, \qquad q^h f_i q^{-h} = q^{-\langle h,\alpha_i\rangle} f_i, \tag{2.2}$$

$$[e_i, f_j] = \delta_{i,j}(t_i - t_i^{-1})/(q_i - q_i^{-1}), \tag{2.3}$$

$$\sum_{k=1}^{1-\langle h_i,\alpha_j\rangle} (-1)^k x_i^{(k)} x_j x_i^{(1-\langle h_i,\alpha_j\rangle-k)} = 0, \qquad (i \neq j) \tag{2.4}$$

where $x = e, f$ and we set $q_i = q^{(\alpha_i,\alpha_i)/2}$, $t_i = q_i^{h_i}$, $[n]_i = (q_i^n - q_i^{-n})/(q_i - q_i^{-1})$, $[n]_i! = \prod_{k=1}^n [k]_i$, $e_i^{(n)} = e_i^n/[n]_i!$ and $f_i^{(n)} = f_i^n/[n]_i!$.

It is well-known that $U_q(\mathfrak{g})$ has a Hopf algebra structure with a comultiplication Δ given by

$$\Delta(q^h) = q^h \otimes q^h, \quad \Delta(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + t_i \otimes f_i,$$

for any $i \in I$ and $h \in P^*$. We do not describe an antipode and a counit. By this comultiplication, a tensor product of $U_q(\mathfrak{g})$-modules has a $U_q(\mathfrak{g})$-module structure.

2.2 Crystals

Let us recall the definition of crystals [8, 9]. The notion of a crystal is motivated by abstracting the some combinatorial properties of crystal bases. We do not write down the definition of crystal base here (See *e.g.* [6, 10, 12]).

Definition 2.1 *A crystal B is a set with the following data:*

$$\text{a map} \quad wt : B \longrightarrow P, \tag{2.5}$$

$$\varepsilon_i : B \longrightarrow \mathbf{Z} \sqcup \{-\infty\}, \quad \varphi_i : B \longrightarrow \mathbf{Z} \sqcup \{-\infty\}, \quad \text{for} \quad i \in I, \tag{2.6}$$

$$\tilde{e}_i : B \longrightarrow B \sqcup \{0\}, \quad \tilde{f}_i : B \longrightarrow B \sqcup \{0\} \quad \text{for} \quad i \in I. \tag{2.7}$$

Here 0 is an ideal element which is not included in B. They are subject to the following axioms: For $b, b_1, b_2 \in B$,

$$\varphi_i(b) = \varepsilon_i(b) + \langle h_i, wt(b)\rangle, \tag{2.8}$$
$$wt(\tilde{e}_i b) = wt(b) + \alpha_i \ \textit{if } \tilde{e}_i b \in B, \tag{2.9}$$
$$wt(\tilde{f}_i b) = wt(b) - \alpha_i \ \textit{if } \tilde{f}_i b \in B, \tag{2.10}$$
$$\tilde{e}_i b_2 = b_1 \ \textit{if and only if } \tilde{f}_i b_1 = b_2, \tag{2.11}$$
$$\textit{if } \varepsilon_i(b) = -\infty, \ \textit{then } \tilde{e}_i b = \tilde{f}_i b = 0. \tag{2.12}$$

From the axiom (2.11), we can consider the graph strucure on a crystal B.

Definition 2.2 *The crystal graph of crystal B is an oriented and colored graph given by the rule : $b_1 \xrightarrow{i} b_2$ if and only if $b_2 = \tilde{f}_i b_1$ $(b_1, b_2 \in B)$.*

Definition 2.3 *(i) If B has the weight decomposition $B = \bigsqcup_{\lambda \in P} B_\lambda$ where $B_\lambda = \{b \in B | wt(b) = \lambda\}$ for $\lambda \in P$, we call B a P-weighted crystal.*

(ii) Let B_1 and B_2 be crystals. A morphism of crystals $\psi : B_1 \longrightarrow B_2$ is a map $\psi : B_1 \sqcup \{0\} \longrightarrow B_2 \sqcup \{0\}$ satisfying the following axioms:

$$\psi(0) = 0, \tag{2.13}$$
$$wt(b) = wt(\psi(b)), \quad \varepsilon_i(b) = \varepsilon_i(\psi(b)), \quad \varphi_i(b) = \varphi_i(\psi(b)) \quad \textit{if } b \in B_1 \textit{ and } \psi(b) \in B_2, \tag{2.14}$$
$$\psi(\tilde{e}_i b) = \tilde{e}_i \psi(b) \ \textit{if } b \in B_1 \textit{ satisfies } \psi(b) \neq 0 \textit{ and } \psi(\tilde{e}_i b) \neq 0, \tag{2.15}$$
$$\psi(\tilde{f}_i b) = \tilde{e}_i \psi(b) \ \textit{if } b \in B_1 \textit{ satisfies } \psi(b) \neq 0 \textit{ and } \psi(\tilde{f}_i b) \neq 0. \tag{2.16}$$

(iii) A morphism of crystals $\psi : B_1 \longrightarrow B_2$ is called strict if the associated map from $B_1 \sqcup \{0\} \longrightarrow B_2 \sqcup \{0\}$ commutes with all $\tilde{e}_i$ and $\tilde{f}_i$. If ψ is injective, surjective and strict, ψ is called an isomorphism.

(iv) A crystal B is a normal, if for any subset J of I such that $((\alpha_i, \alpha_j))_{i,j \in J}$ is a positive symmetric matrix, B is isomorphic to a crystal base of an integrable $U_q(\mathfrak{g}_J)$-module, where $U_q(\mathfrak{g}_J)$ is the quantum algebra generated by e_j, f_j $(j \in J)$ and q^h $(h \in P^*)$.

For crystals B_1 and B_2, we shall define their tensor product $B_1 \otimes B_2$ as follows:

$$B_1 \otimes B_2 = \{b_1 \otimes b_2 | b_1 \in B_1, \, b_2 \in B_2\}, \tag{2.17}$$
$$wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2), \tag{2.18}$$
$$\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, wt(b_1)\rangle), \tag{2.19}$$
$$\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, wt(b_2)\rangle), \tag{2.20}$$

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \tag{2.21}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases} \tag{2.22}$$

Here we understand that $0 \otimes b = b \otimes 0 = 0$. Let $\mathcal{C}(I, P)$ be the category of crystals determined by the data I and P. Then $\otimes$ is a functor from $\mathcal{C}(I, P) \times \mathcal{C}(I, P)$ to $\mathcal{C}(I, P)$ and satisfies the associative law: $(B_1 \otimes B_2) \otimes B_3 \cong B_1 \otimes (B_2 \otimes B_3)$ by $(b_1 \otimes b_2) \otimes b_3 \leftrightarrow b_1 \otimes (b_2 \otimes b_3)$. Therefore, the category of crystals is endowed with the structure of tensor category.

Example 2.4 *We give some examples of crystals.*

(i) $C = \{c\}$ *with*

$$wt(c) = 0, \quad \varepsilon_i(c) = \varphi_i(c) = 0, \quad \tilde{e}_i c = \tilde{f}_i c = 0.$$

This is isomorphic to the crystal of the trivial $U_q(\mathfrak{g})$*-module.*

(ii) For $\lambda \in P$*, we set* $T_\lambda = \{t_\lambda\}$ *with*

$$wt(t_\lambda) = \lambda, \quad \varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty, \quad \tilde{e}_i(t_\lambda) = \tilde{f}_i(t_\lambda) = 0.$$

We can see that $T_\lambda \otimes T_\mu \cong T_{\lambda+\mu}$ *and* $B \otimes T_0 \cong T_0 \otimes B \cong B$ *for any crystal* B*.*

(iii) For $\lambda \in P_+$*, let* $(L(\lambda), B(\lambda))$ *be the crystal base of a* $U_q(\mathfrak{g})$*-integrable highest weight module* $V(\lambda)$ *([8]).* $B(\lambda)$ *is the crystal associated with* $V(\lambda)$*. Let* $B(-\lambda)$ *be the crystal associated with a integrable lowest weight module* $V(-\lambda)$*. Then* C *is isomorphic to* $B(0)$*.*

(iv) Let $(L(\infty), B(\infty))$ *be a crystal base of* $U_q^-(\mathfrak{g})$ *([8]).* $B(\infty)$ *is a crystal associated with* $U_q^-(\mathfrak{g})$*. We shall also denote* $B(-\infty)$ *for a crystal associated with* $U_q^+(\mathfrak{g})$*.*

3 Crystals of modified quantum algebra

This section is devoted to review [9],[14] (See also [13]).

3.1 Modified quantum algebra and Crystal base

For an integral weight $\lambda \in P$, let $U_q(\mathfrak{g})a_\lambda$ be the left $U_q(\mathfrak{g})$-module given by

$$U_q(\mathfrak{g})a_\lambda := U_q(\mathfrak{g}) / \sum_{h \in P^*} U_q(\mathfrak{g})(q^h - q^{\langle h, \lambda \rangle}), \tag{3.1}$$

where a_λ is the image of the unit by the canonical projection. We set

$$\widetilde{U}_q(\mathfrak{g}) = \bigoplus_{\lambda \in P} U_q(\mathfrak{g}) a_\lambda, \tag{3.2}$$

which is called *modified quantum algebra*.

We shall see a crystal base of $\widetilde{U}_q(\mathfrak{g})$. Taking $\lambda \in P$ and choosing $\zeta, \mu \in P_+$ such that $\lambda = \zeta - \mu$, we get the following $U_q(\mathfrak{g})$-linear surjective homomorphism:

$$\begin{array}{rccl} \pi_{\zeta,\mu} : U_q(\mathfrak{g}) a_\lambda & \longrightarrow & V(\zeta) \otimes V(-\mu), \\ a_\lambda & \mapsto & u_\zeta \otimes u_{-\mu}. \end{array} \tag{3.3}$$

where $V(\zeta)$ and $V(-\mu)$ are as in Example 2.4 (iii) and u_ζ and $u_{-\mu}$ are their highest weight vector and lowest weight vector respectively.

Theorem 3.1 (*cf* **[14]**) *For any $\lambda \in P$, there exists a unique pair $(L(U_q(\mathfrak{g})a_\lambda), B(U_q(\mathfrak{g})a_\lambda))$ which satisfies the following properties.*

(i) *We set $A := \{f(q) \in \mathbf{Q}(q) | f$ has no pole at $q = 0\}$. $L(U_q(\mathfrak{g})a_\lambda)$ is a free A-module such that $U_q(\mathfrak{g})a_\lambda \cong \mathbf{Q}(q) \otimes_A L(U_q(\mathfrak{g})a_\lambda)$ and $B(U_q(\mathfrak{g})a_\lambda)$ is a $\mathbf{Q}$-basis of the $\mathbf{Q}$-vector space $L(U_q(\mathfrak{g})a_\lambda)/qL(U_q(\mathfrak{g})a_\lambda)$.*

(ii) *For any $\zeta, \mu \in P_+$ with $\lambda = \zeta - \mu$, we have*

$$\pi_{\zeta,\mu}(L(U_q(\mathfrak{g})a_\lambda)) \subset L(\zeta) \otimes_A L(-\mu),$$

and the induced map $\bar{\pi}_{\zeta,\mu}$:

$$\bar{\pi}_{\zeta,\mu} \; : \; L(U_q(\mathfrak{g})a_\lambda)/qL(U_q(\mathfrak{g})a_\lambda) \longrightarrow (L(\zeta)/qL(\zeta)) \otimes (L(-\mu)/qL(-\mu)),$$

satisfies $\bar{\pi}_{\zeta,\mu}(B(U_q(\mathfrak{g})a_\lambda)) \subset B(\zeta) \otimes B(-\mu) \sqcup \{0\}$.

(iii) *There is a structure of crystal on $B(U_q(\mathfrak{g})a_\lambda)$ such that $\bar{\pi}_{\zeta,\mu}$ gives a strict morphism of crystals for any $\zeta, \mu \in P_+$ with $\lambda = \zeta - \mu$.*

Set

$$(L(\widetilde{U}_q(\mathfrak{g})), B(\widetilde{U}_q(\mathfrak{g}))) := \bigoplus_{\lambda \in P} (L(U_q(\mathfrak{g})a_\lambda), B(U_q(\mathfrak{g})a_\lambda)).$$

Remark. $B(U_q(\mathfrak{g})a_\lambda)$ is a normal crystal and then $B(\widetilde{U}_q(\mathfrak{g}))$ is a normal crystal.

Let $B(\infty)$, $B(-\infty)$ and T_λ ($\lambda \in P$) be the crystals given in Example 2.4. The following theorem plays a significant role in this paper (See [9, Sec.3]).

Theorem 3.2 $B(U_q(\mathfrak{g})a_\lambda) \cong B(\infty) \otimes T_\lambda \otimes B(-\infty)$ *as a crystal.*

Corollary 3.3 $B(\widetilde{U}_q(\mathfrak{g})) \cong \oplus_{\lambda \in P} B(\infty) \otimes T_\lambda \otimes B(-\infty)$ *as a crystal.*

3.2 Weyl group action and Extremal vectors

This subsection is devoted to review [9, Sec.7.8.9]. Let B be a normal crystal (See 2.2). Let us define the Weyl group action on the underlying set B. For $i \in I$ and $b \in B$, we set

$$S_i b = \begin{cases} \tilde{f}_i^{\langle h_i, wt(b)\rangle} b & \text{if } \langle h_i, wt(b)\rangle \geq 0 \\ \tilde{e}_i^{-\langle h_i, wt(b)\rangle} b & \text{if } \langle h_i, wt(b)\rangle < 0. \end{cases} \tag{3.4}$$

We can easily obtain the following formula:

$$S_i^2 = \mathrm{id}, \qquad S_i \tilde{e}_i = \tilde{f}_i S_i, \qquad wt(S_i b) = s_i(wt(b)),$$

where $s_i(\lambda) = \lambda - \langle h_i, \lambda\rangle \alpha_i$ is the simple reflection.

Let $\mathfrak{g}$ be a rank 2 finite dimensional Lie algebra, and W be the Weyl group associated with $\mathfrak{g}$. We set $w_0 = s_{i_1} \cdots s_{i_k}$ a reduced expression of the longest element of W. Here we get the following ([9, Sec.7]):

Proposition 3.4 *Let B be a normal crystal. For any $b \in B$, $S_{i_1} \cdots S_{i_k} b$ does not depend on the choice of reduced expression.*

Corollary 3.5 *$\{S_i\}_{i=1,2}$ satisfies the braid relation.*

Thus for general $\mathfrak{g}$, we know that $\{S_i\}_{i \in I}$ defines the Weyl group action on a normal crystal B.

Definition 3.6 *(i) Let B be a normal crystal. An element $b \in B$ is called i-extremal, if $\tilde{e}_i b = 0$ or $\tilde{f}_i b = 0$.*

(ii) An element $b \in B$ is called extremal if for any $l \geq 0$, $S_{i_1} \cdots S_{i_l} b$ is i-extremal for any i, $i_1 \cdots i_l \in I$.

The following theorems play a significant role in Sec.7.

Theorem 3.7 *Any connected component of $B(\widetilde{U}_q(\mathfrak{g}))$ contains an extremal vector.*

4 Affine crystals

In the rest of this paper, we fix $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$. Let us denote U for $U_q(\widehat{\mathfrak{sl}_2})$.

4.1 Notations

We follow the notations in [3], [4]. Let Λ_0 and Λ_1 be fundamental weights and δ a generator of null roots. Then we can write a weight lattice $P = \mathbf{Z}\Lambda_0 \oplus \mathbf{Z}\Lambda_1 \oplus \mathbf{Z}\delta$, its dual lattice $P^* = \mathbf{Z}h_0 \oplus \mathbf{Z}h_1 \oplus \mathbf{Z}d$ and the Cartan subalgebra $\mathfrak{t} = \mathbf{Q}h_0 \oplus \mathbf{Q}h_1 \oplus \mathbf{Q}d$, where $\langle h_i, \Lambda_j \rangle = \delta_{ij}$, $\langle h_i, \delta \rangle = \langle d, \Lambda_i \rangle = 0$, $\langle d, \delta \rangle = 1$ $\alpha_1 = 2\Lambda_1 - 2\Lambda_0$ and $\alpha_0 = \delta - \alpha_1$. Set $P_{cl} = P/\mathbf{Z}\delta$ and let $cl : P \longrightarrow P_{cl}$ be a canonical projection. Then we set $(P_{cl})^* = \mathbf{Z}h_0 \oplus \mathbf{Z}h_1$. Now, we fix a map $af : P_{cl} \longrightarrow P$ such that $af \circ cl(\Lambda_i) = \Lambda_i$. Here note that $af \circ cl(\alpha_1) = \alpha_1$ and $al \circ cl(\alpha_0) = \alpha_0 - \delta$. In this specific case, we call an element of P an *affine weight* and an element of P_{cl} a *classical weight*. In the rest of this paper, if there is no confusion, we shall denote Λ_i for a classical weight $cl(\Lambda_i)$.

4.2 Affinization of classical crystals

U is the quantized enveloping algebra associated with P. Let U' be its subalgebra generated by e_i, f_i and $q^h \quad (h \in (P_{cl})^*)$. The algebra U' is also a quantized enveloping algebra associated with P_{cl}. Now, we call a P-weighted crystal an *affine crystal* and a P_{cl}-weighted crystal a *classical crystal*.

Remark.

(i) The simple roots of $U' : \{cl(\alpha_i)\}$ are not linearly independent.

(ii) A U-module has a U'-module structure but in general, the opposite case is false.

Definition 4.1 *Let B be a classical crystal. We define the affine crystal* $\mathrm{Aff}(B)$ *associated with B as follows;*

$$\mathrm{Aff}(B) := \{z^n \otimes b \,|\, b \in B,\, n \in \mathbf{Z}\}, \tag{4.1}$$

where z is an indeterminate. We call $\mathrm{Aff}(B)$ *an affinization of B. The actions by $\tilde{e}_i$ and $\tilde{f}_i$, and the data are given as follows:*

$$\begin{aligned} &\tilde{e}_i(z^n \otimes b) = z^{n+\delta_{i,0}} \otimes \tilde{e}_i(b), \quad \tilde{f}_i(z^n \otimes b) = z^{n-\delta_{i,0}} \otimes \tilde{f}_i(b), \\ &\varepsilon_i(z^n \otimes b) = \varepsilon_i(b), \quad \varphi_i(z^n \otimes b) = \varphi_i(b), \quad wt(z^n \otimes b) = n\delta + af(wt(b)). \end{aligned} \tag{4.2}$$

Here, note that even if a classical crystal B is connected as a crystal graph, its affinization $\mathrm{Aff}(B)$ is not necessarily connected.

Example 4.2 *Let $B = \{+, -\}$ be a 2-dimensional classical crystal given by*

$$\begin{aligned} &\tilde{e}_0(+) = \tilde{f}_1(+) = -, \quad \tilde{e}_1(-) = \tilde{f}_0(-) = +, \\ &\tilde{e}_0(-) = \tilde{f}_1(-) = 0, \quad \tilde{e}_1(+) = \tilde{f}_0(+) = 0, \\ &\varphi_1(+) = \varphi_0(-) = \varepsilon_0(+) = \varepsilon_1(-) = 1, \\ &\varphi_1(-) = \varphi_0(+) = \varepsilon_0(-) = \varepsilon_1(+) = 0, \\ &wt(\pm) = \pm(\Lambda_1 - \Lambda_0). \end{aligned} \tag{4.3}$$

It is easy to see that $B^{\otimes 2}$ is connected. But its affinization

$$\mathrm{Aff}(B^{\otimes 2}) \cong \{z^n \otimes \epsilon_1 \otimes \epsilon_2 | \epsilon_i = \pm, \quad n \in \mathbf{Z}\} \tag{4.4}$$

is not connected. In fact, this is divided into the following two components;

$$\begin{aligned}
\mathrm{Aff}(B^{\otimes 2})_1 &:= \{z^{2n-1} \otimes + \otimes +,\ z^{2n-1} \otimes - \otimes +,\ z^{2n-1} \otimes - \otimes -,\ z^{2n} \otimes + \otimes -, | n \in \mathbf{Z}\}.\\
\mathrm{Aff}(B^{\otimes 2})_0 &:= \{z^{2n} \otimes + \otimes +,\ z^{2n} \otimes - \otimes +,\ z^{2n} \otimes - \otimes -,\ z^{2n+1} \otimes + \otimes -, | n \in \mathbf{Z}\},
\end{aligned}$$

4.3 Extremal vectors in $B^{\otimes n}$

In this subsection, let B be the 2-dimensional classical crystal introduced in Example 4.2.

Now, we classify all extremal vectors in $B^{\otimes n}$.

Proposition 4.3 *$B^{\otimes n}$ is connected as a crystal graph. Let E be the set of all extremal vectors in $B^{\otimes n}$. Then we have*

$$E = \{(+)^{\otimes n}, (-)^{\otimes n}\}.$$

Proof. By the fact that B is a perfect crystal [4, Corollary 4.6.3.], we can easily obtain the connectedness of $B^{\otimes n}$.

By [12, Sec.2], we know that

$$\begin{aligned}
\tilde{f}_1((-)^{\otimes k} \otimes (+)^{\otimes l}) &= (-)^{\otimes k+1} \otimes (+)^{\otimes l-1},\\
\tilde{e}_1((-)^{\otimes k} \otimes (+)^{\otimes l}) &= (-)^{\otimes k-1} \otimes (+)^{\otimes l+1},\\
\tilde{f}_0((+)^{\otimes k} \otimes (-)^{\otimes l}) &= (+)^{\otimes k+1} \otimes (-)^{\otimes l-1},\\
\tilde{e}_0((+)^{\otimes k} \otimes (-)^{\otimes l}) &= (+)^{\otimes k-1} \otimes (-)^{\otimes l+1},
\end{aligned} \tag{4.5}$$

where we consider $(\pm)^{\otimes m} = 0$ if $m < 0$. Since $B^{\otimes n}$ is a normal crystal, we get

$$S_1(+)^{\otimes n} = S_0(+)^{\otimes n} = (-)^{\otimes n}, \quad S_1(-)^{\otimes n} = S_0(-)^{\otimes n} = (+)^{\otimes n}. \tag{4.6}$$

From (4.6) and the fact that $\tilde{e}_1(+)^{\otimes n} = \tilde{e}_0(-)^{\otimes n} = \tilde{f}_1(-)^{\otimes n} = \tilde{f}_0(+)^{\otimes n} = 0$, we know that $(+)^{\otimes n}$ and $(-)^{\otimes n}$ are extremal vectors.

Now we shall show that there is no other extremal vector without these two vectors by using the induction on n.

For $n = 1$, this is trivial. We assume that $u \otimes +$ is an extremal vector in $B^{\otimes n+1}$. Here note that for any $b \in B^{\otimes n}$, $\varphi_i(b)$ and $\varepsilon_i(b)$ are given by

$$\begin{aligned}
\varphi_i(b) &= \max\{n; \tilde{f}_i^n b \neq 0\},\\
\varepsilon_i(b) &= \max\{n; \tilde{e}_i^n b \neq 0\},
\end{aligned}$$

and then

$$\varphi_i(b) \geq 0, \quad \varepsilon_i(b) \geq 0. \tag{4.7}$$

We have $\tilde{f}_1(u \otimes +) \neq 0$ by $\tilde{f}_1(+) \neq 0$ and (2.22). Then, by the definition of extremal vectors, we have $\tilde{e}_1(u \otimes +) = 0$ and then

$$\tilde{e}_1 u = 0, \tag{4.8}$$

since we have that $\varepsilon_1(+) = 0$, $\varphi_1(u) \geq 0$ and then $\tilde{e}_1(u \otimes +) = \tilde{e}_1(u) \otimes +$ by (2.21).
We shall show

$$\tilde{e}_0(u \otimes +) \neq 0. \tag{4.9}$$

If $\varphi_0(u) \geq 1$, $\varphi_0(u) \geq 1 = \varepsilon_0(+)$ and then $\tilde{e}_0(u \otimes +) = \tilde{e}_0(u) \otimes + \neq 0$. Otherwise, $\varphi_0(u) < \varepsilon_0(+)$ and then $\tilde{e}_0(u \otimes +) = u \otimes \tilde{e}_0(+) = u \otimes - \neq 0$. We get (4.9). By the definition of extremal vector, we have

$$\tilde{f}_0(u \otimes +) = 0. \tag{4.10}$$

By (4.8), we have $\varepsilon_1(u) = 0$. Then we get $\langle h_1, wt(u \otimes +)\rangle = \langle h_1, wt(u)\rangle + 1 = \varphi_1(u) + 1 > 0$. Thus

$$\begin{aligned} S_1(u \otimes +) &= \tilde{f}_1^{\langle h_1, wt(u\otimes +)\rangle}(u \otimes +) = \tilde{f}_1^{\varphi_1(u)} u \otimes \tilde{f}_1(+) \\ &= S_1(u) \otimes -. \end{aligned}$$

This $S_1(u) \otimes -$ is extremal and $\tilde{e}_1(u \otimes +) \neq 0$. Therefore, by the similar argument as above, we get $\tilde{e}_0 S_1(u) \neq 0$ and

$$S_0(S_1(u) \otimes -) = S_0 S_1(u) \otimes +. \tag{4.11}$$

By arguing similarly, we get $\tilde{e}_1(S_0 S_1 u) = 0$. Repeating these argument, we obtain

$$\tilde{e}_1(S_0 S_1 \cdots S_1 u) = 0,\ \tilde{e}_0(S_1 S_0 \cdots S_1 u) = 0. \tag{4.12}$$

The set $\{(S_0 S_1)^k u\}_{k \in \mathbf{Z}_{\geq 0}}$ is a subset of the finite set $B^{\otimes n}$. Then there exist $l,\ m \in \mathbf{Z}_{\geq 0}$ such that $l > m$,

$$(S_0 S_1)^l u = (S_0 S_1)^m u.$$

Then we have $(S_0 S_1)^{l-m} u = u$ $(l - m > 0)$ and then for any $p \geq 0$ there exists $r \in \mathbf{Z}_{>0}$ such that $(l-m)r > p$. Thus we have

$$(S_1 S_0)^p u = (S_0 S_1)^{(l-m)r-p} u,\ S_0(S_1 S_0)^p u = S_1(S_0 S_1)^{(l-m)r-p-1} u. \tag{4.13}$$

By (4.12) and (4.13), we obtain

$$\tilde{e}_1(S_1 S_0 \cdots S_1 S_0) u = 0,\ \tilde{e}_0(S_0 S_1 \cdots S_1 S_0) u = 0. \tag{4.14}$$

We shall show

$$\tilde{f}_0 u = 0. \tag{4.15}$$

Assuming $\tilde{f}_0 u \neq 0$, we shall derive a contradiction. The assumption implies $\varphi_0(u) > 0$. If $\varphi_0(u) \geq 2$, $\tilde{f}_0(u \otimes +) = \tilde{f}_0(u) \otimes + \neq 0$ since $\varepsilon_0(+) = 1$. This contradicts (4.10). Then we know that

$$\varphi_0(u) = 1. \tag{4.16}$$

Now we write $u = u_1 \otimes u_2 \otimes \cdots \otimes u_n$ ($u_j = \pm$). By using (4.8) and (4.16), we get

$$u_1 = +, \tag{4.17}$$

$$\varepsilon_0(u) \geq \varphi_0(u) = 1, \tag{4.18}$$

because $0 \leq \langle h_1, wt(u) \rangle = -\langle h_0, wt(u) \rangle = \varepsilon_0(u) - \varphi_0(u)$. Now, by applying Remark 2.1.2 in [12], (4.17) and (4.18) to $S_0\, u$. It is easily obtained that $S_0 u = \tilde{e}_0^{-\langle h_0, wt(u) \rangle} u = \tilde{e}_0^{\varepsilon_0(u)-1} u$ is in the following form:

$$S_0\, u = + \otimes u', \tag{4.19}$$

where $u' \in B^{\otimes n-1}$, *i.e.* the action of S_0 never touches u_1. A vector in the form (4.19) does not vanish by the action of $\tilde{e}_0$. This contradicts (4.14) and then we get $\varphi_0(u) = 0$.
We have

$$\begin{aligned} S_0(u \otimes +) &= \tilde{e}_0^{-\langle h_0, wt(u \otimes +) \rangle}(u \otimes +) = \tilde{e}_0^{\varepsilon_0(u) - \varphi_0(u) + 1}(u \otimes +) \\ &= \tilde{e}_0^{\varepsilon_0(u)+1}(u \otimes +) = \tilde{e}_0^{\varepsilon_0(u)}\, u \otimes \tilde{e}_0(+) \\ &= S_0\, u \otimes - \end{aligned}$$

The vector $S_0\, u \otimes -$ does not vanish by the action of $\tilde{f}_0$ since $\tilde{f}_0(-) \neq 0$. Since $S_0 u \otimes -$ is an extremal vector, this vanishes by the action of $\tilde{e}_0$. By the similar argument to obtain (4.9), we have $\tilde{e}_1(S_0\, u \otimes -) \neq 0$. Then

$$\tilde{f}_1(S_0\, u \otimes -) = 0. \tag{4.20}$$

By exchanging $+$ and $-$, and arguing similarly to get (4.15), we get

$$\tilde{f}_1(S_0\, u) = 0. \tag{4.21}$$

By repeating above argument, we obtain

$$\tilde{f}_0(S_1 S_0 \cdots S_0\, u) = 0, \;\; \tilde{f}_1(S_0 S_1 \cdots S_0\, u) = 0. \tag{4.22}$$

Furthermore, by the similar argument to get (4.14), we get

$$\tilde{f}_0(S_0 S_1 \cdots S_1\, u) = 0, \;\; \tilde{f}_1(S_1 S_0 \cdots S_1\, u) = 0. \tag{4.23}$$

By (4.12), (4.14), (4.22) and (4.23), we know that the vector u is an extremal vector in $B^{\otimes n}$. By the hypothesis of the induction and (4.17), we get

$$u = (+)^{\otimes n} \;\text{ and then }\; u \otimes + = (+)^{\otimes n+1}.$$

By assuming $u \otimes -$ is an extremal vector in $B^{\otimes n+1}$ and discussing similarly, we get $u = (-)^{\otimes n}$ and then $u \otimes - = (-)^{\otimes n+1}$. □

5 Path realization for $B(Ua_\lambda)$ with level of $\lambda = 0$

As in the previous section, we set $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$ in this section.

5.1 Crystal $B(\infty)$ and $B(-\infty)$

Now, we define the following $\widehat{\mathfrak{sl}_2}$-classical crystal:

Definition 5.1 *We set*

$$B_\infty := \{(n) | n \in \mathbf{Z}\}, \quad (wt(n) = 2n(\Lambda_0 - \Lambda_1))$$

and define the actions of $\tilde{e}_i$ and $\tilde{f}_i$ as follows;

$$\tilde{e}_1(n) = (n-1),\ \tilde{f}_1(n) = (n+1),\ \tilde{e}_0(n) = (n+1),\ \tilde{f}_0(n) = (n-1).$$

We also set

$$\varepsilon_1(n) = n,\ \varphi_1(n) = -n,\ \varepsilon_0(n) = -n,\ \varphi_0(n) = n.$$

By the above data, B_∞ is equipped with a classical crystal structure.

We introduce the following remarkable result (see [11]).

Proposition 5.2 *Let B_∞ be as above. We get the following isomorphism of classical crystal:*

$$\begin{array}{ccc} B(\infty) & \xrightarrow{\sim} & B(\infty) \otimes B_\infty \quad (\text{ resp. } B(-\infty) \xrightarrow{\sim} B_\infty \otimes B(-\infty)), \\ u_\infty & \mapsto & u_\infty \otimes (0) \quad (\text{resp. } u_{-\infty} \mapsto (0) \otimes u_{-\infty}). \end{array} \tag{5.1}$$

By applying this proposition repeatedly, we get for any $k > 0$,

$$\begin{array}{ccc} \psi_k : B(\infty) & \xrightarrow{\sim} & B(\infty) \otimes B_\infty^{\otimes k} \, (\text{ resp. } B(-\infty) \xrightarrow{\sim} B_\infty^{\otimes k} \otimes B(-\infty)), \\ u_\infty & \mapsto & u_\infty \otimes (0)^{\otimes k} \, (\text{resp. } u_{-\infty} \mapsto (0)^{\otimes k} \otimes u_{-\infty}). \end{array} \tag{5.2}$$

Lemma 5.3 *For any $b \in B(\infty)$ (resp. $B(-\infty)$), there exists $k > 0$ such that*

$$\psi_k(b) \in u_\infty \otimes B_\infty^{\otimes k} \quad (\text{ resp. } B_\infty^{\otimes k} \otimes u_{-\infty}). \tag{5.3}$$

We set

$$\mathcal{P}(\infty) := \{(\cdot\cdot, i_k, i_{k+1}, \cdot\cdot, i_{-1}) | i_k \in B_\infty \text{ and } if |k| \gg 0, i_k = (0)\}, \tag{5.4}$$

$$\mathcal{P}(-\infty) := \{(i_0, \cdot\cdot, i_k, i_{k+1}, \cdot\cdot) | i_k \in B_\infty \text{ and } if |k| \gg 0, i_k = (0)\}, \tag{5.5}$$

Now, we consider formally $u_\infty = \cdots \otimes (0) \otimes (0) = (\cdots, (0), (0))$ (resp. $u_{-\infty} = (0) \otimes (0) \otimes \cdots = ((0), (0), \cdots)$). Then by (5.2) and Lemma 5.3, we get the following isomorphism between $B(\infty)$ (resp. $B(-\infty)$) and $\mathcal{P}(\infty)$ (resp. $\mathcal{P}(-\infty)$).

Proposition 5.4 *The crystal $B(\infty)$ (resp. $B(-\infty)$) is isomorphic to $\mathcal{P}(\infty)$ (resp. $\mathcal{P}(-\infty)$) given by $B(\infty) \ni b \mapsto p \in \mathcal{P}(\infty)$ (resp. $B(-\infty) \ni b \mapsto p \in \mathcal{P}(-\infty)$) where $\psi_k(b) = u_\infty \otimes i_k \otimes \cdots \otimes i_{-2} \otimes i_{-1}$ (resp. $\psi_k(b) = i_0 \otimes i_1 \otimes \cdots \otimes i_k \otimes u_{-\infty}$) for $|k| \gg 0$.*

5.2 Path

Lemma 5.5 *We set* $\lambda = m(\Lambda_0 - \Lambda_1) \in P_{cl}$ $(m \in \mathbf{Z})$*. Then the map*

$$\begin{array}{rll} \varphi : & T_\lambda \otimes B_\infty & \xrightarrow{\sim} B_\infty \otimes T_{-\lambda}, \\ & t_\lambda \otimes (n) & \mapsto (m+n) \otimes t_{-\lambda}, \end{array} \tag{5.6}$$

is an isomorphism between classical crystals.

Proof. It is trivial that the map φ is bijective. Then we shall show that φ is a strict morphism of classical crystals (See Definition 2.3). The weight of $t_\lambda \otimes (n)$ is $(m+2n)(\Lambda_0 - \Lambda_1)$ and the one of $(m+n) \otimes t_{-\lambda}$ is also $(m+2n)(\Lambda_0 - \Lambda_1)$. Thus, φ preseves weights of crystals. By Example 2.4 (ii), we can see that $\tilde{e}_i(t_\lambda \otimes (n)) = t_\lambda \otimes \tilde{e}_i(n)$, $\tilde{f}_i(t_\lambda \otimes (n)) = t_\lambda \otimes \tilde{f}_i(n)$, $\tilde{e}_i((m+n) \otimes t_{-\lambda}) = \tilde{e}_i(m+n) \otimes t_{-\lambda}$, $\tilde{f}_i((m+n) \otimes t_{-\lambda}) = \tilde{f}_i(m+n) \otimes t_{-\lambda}$. Thus we get $\varphi\tilde{e}_i = \tilde{e}_i\varphi$ and $\varphi\tilde{f}_i = \tilde{f}_i\varphi$. Now, it is remaining to show that $\varepsilon_i(t_\lambda \otimes (n)) = \varepsilon_i((m+n) \otimes t_{-\lambda})$ and $\varphi_i(t_\lambda \otimes (n)) = \varphi_i((m+n) \otimes t_{-\lambda})$. In order to show these, it is enough to notice the following lemma.

Lemma 5.6 *Let* B *be a crystal. For* $b \in B$ *and* λ, $\mu \in P$*, we get*

$$\begin{aligned} \varepsilon_i(t_\lambda \otimes b \otimes t_\mu) &= \varepsilon_i(b) - \langle h_i, \lambda \rangle, \\ \varphi_i(t_\lambda \otimes b \otimes t_\mu) &= \varphi_i(b) + \langle h_i, \mu \rangle. \end{aligned}$$

This lemma is trivial from Example 2.4 (ii), (2.21) and (2.22). Therefore, we obtain the following formula:

$$\begin{aligned} \varepsilon_1(t_\lambda \otimes (n)) &= m + n = \varepsilon_1((m+n) \otimes t_{-\lambda}), \\ \varepsilon_0(t_\lambda \otimes (n)) &= -m - n = \varepsilon_0((m+n) \otimes t_{-\lambda}), \\ \varphi_1(t_\lambda \otimes (n)) &= -n = \varphi_1((m+n) \otimes t_{-\lambda}), \\ \varphi_0(t_\lambda \otimes (n)) &= n = \varphi_0((m+n) \otimes t_{-\lambda}). \end{aligned}$$

We completed the proof. □

Applying

$$T_\lambda \otimes T_{-\lambda} \cong T_{-\lambda} \otimes T_\lambda \cong T_0, \tag{5.7}$$

$$B \otimes T_0 \cong T_0 \otimes B \cong B, \tag{5.8}$$

to (5.6), we get isomorphisms:

$$\begin{array}{rcl} \varphi_- : B_\infty & \xrightarrow{\sim} & T_{-\lambda} \otimes B_\infty \otimes T_{-\lambda}, \\ (n) & \mapsto & t_{-\lambda} \otimes (m+n) \otimes t_{-\lambda}, \end{array} \tag{5.9}$$

$$\begin{array}{rcl} \varphi_+ : B_\infty & \xrightarrow{\sim} & T_\lambda \otimes B_\infty \otimes T_\lambda, \\ (n) & \mapsto & t_\lambda \otimes (n-m) \otimes t_\lambda. \end{array} \tag{5.10}$$

By applying (5.9) and (5.10) to (5.1), we get the following isomorphisms of crystal:

$$\begin{array}{rcll} B(-\infty) & \xrightarrow{\sim} & T_{-\lambda} \otimes B_\infty \otimes T_{-\lambda} \otimes B(-\infty), & (5.11) \\ u_{-\infty} & \mapsto & t_{-\lambda} \otimes (m) \otimes t_{-\lambda} \otimes u_{-\infty}, & \\ B(-\infty) & \xrightarrow{\sim} & T_{\lambda} \otimes B_\infty \otimes T_{\lambda} \otimes B(-\infty), & (5.12) \\ u_{-\infty} & \mapsto & t_{\lambda} \otimes (-m) \otimes t_{\lambda} \otimes u_{-\infty}. & \end{array}$$

By combining (5.11) and (5.12), and using (5.7) and (5.8) again, we obtain an isomorphism of crystal,

$$\begin{array}{rcl} T_\lambda \otimes B(-\infty) & \xrightarrow{\sim} & B_\infty \otimes B_\infty \otimes T_\lambda \otimes B(-\infty) \\ t_\lambda \otimes u_{-\infty} & \mapsto & (m) \otimes (-m) \otimes t_\lambda \otimes u_{-\infty}. \end{array} \tag{5.13}$$

Now, we set

$\mathcal{P}_m(-\infty) := \{p = (i_0, i_1, \cdot\cdot, i_k, \cdot\cdot) | i_k \in B_\infty$ and if $|k| \gg 0, i_{2k} = (m)$ and $i_{2k+1} = (-m)\}$.

By using (5.13) repeatedly and arguing similarly as in 5.1, we get

Proposition 5.7 *The following is an isomorphism of crystal;*

$$T_\lambda \otimes B(-\infty) \cong \mathcal{P}_m(-\infty). \tag{5.14}$$

Here note that $t_\lambda \otimes u_{-\infty} \mapsto (m) \otimes (-m) \otimes (m) \otimes (-m) \otimes \cdots$.

We set

$$\begin{array}{c} \mathcal{P}_m := \{p = (\cdots, i_k, i_{k+1}, \cdots, i_{-1}, i_0, i_1, \cdots, i_l, i_{l+1}, \cdots) |\ i_k \in B_\infty \\ \text{if } k \ll 0,\ i_k = (0) \text{ and if } l \gg 0,\ i_{2l} = (m) \text{ and } i_{2l+1} = (-m)\}. \end{array} \tag{5.15}$$

Now let us call an element of $\mathcal{P}_m$ *m-path* or simply, *path*.

By applying (5.4) and (5.14) to Theorem 3.2, we can easily obtain the following result:

Theorem 5.8 *For* $\lambda = m(\Lambda_0 - \Lambda_1) \in P_{cl}$ $(m \in \mathbf{Z})$,

$$B(U' a_\lambda) \cong \mathcal{P}_m. \tag{5.16}$$

Here note that this is an isomorphism of classical crystals.

5.3 Wall and Domain

For this subsection, see *e.g.* [2], [3]. In the rest of this paper, we identify B_∞ with $\mathbf{Z}$. Thus, for $(i) \in B_\infty$ we denote i and then for $i, j \in B_\infty$ we can formally consider the summation and subtraction $i \pm j$, and the absolute value $|i|$.

We fix an integer $m \in \mathbf{Z}$ and let $p \in \mathcal{P}_m$ be a m-path.

Definition 5.9 (*i*) *A path* $p = (\cdots, i_{k-1}, i_k, \cdots)$ *has* l *walls in the position* k ($l \in \mathbf{Z}_{>0}$, $k \in \mathbf{Z}$), *if* $|i_{k-1} + i_k| = l$.

(*ii*) *Suppose that there are walls in position* k. *The type of walls in position* k *is* $+$ (*resp.* $-$) *if* $i_{k-1} + i_k > 0$ (*resp.* $i_{k-1} + i_k < 0$).

We also define a function $n : \mathcal{P}_m \longrightarrow \mathbf{Z}_{\geq 0}$ by

$$n(p) = \sum_{k \in \mathbf{Z}} |i_{k-1} + i_k|$$

and we call this the total number of walls in p.

Here note that for any $p \in \mathcal{P}_m$, $n(p)$ is finite by the definition of $\mathcal{P}_m$.

Definition 5.10 *A segment* $d = i_j, i_{j+1}, \cdots, i_l$ *in* $p \in \mathcal{P}_m$ *is a finite domain with length* $l - j + 1$ *in* p *if there are walls in the position* j *and* $l + 1$ *and there is no wall in positions* $j + 1, j + 2, \cdots, l$. *We denote* $l(d) := l - j + 1$ *for the length of domain* d.

Remark.

(*i*) In this definition, we can consider a domain with length 0. This occurs in the following case. If there are more than two walls in the same position, there is a domain with length 0 between a pair of neighboring two walls.

(*ii*) By the definition of $\mathcal{P}_m$, we know that any path has two infinite sequences in the forms $\cdots 0, 0, 0, 0$ and $\pm m, \mp m, \pm m, \cdots$. We call these *infinite domains.*

(*iii*) By the definition of finite domain, any finite domain with positive length is in the following form;

$$k, -k, k, -k, \cdots, \pm k, \mp k. \tag{5.17}$$

Example 5.11 *For* $p = (\cdots, 0, 0, 1, -1, 3, -3, 3 \cdots)$, *we visualize walls and domains:*

$$\cdots 00|1 - 1||3 - 33 \cdots. \tag{5.18}$$

In (5.18), we know that there are three walls, two finite domains : $1 - 1$ *and a zero-length domain and two infinite domains* : $\cdots 00$ *and* $3 - 33 \cdots$.

Now, for $n \in \mathbf{Z}_{\geq 0}$ we set

$$\mathcal{P}_m(n) := \{p \in \mathcal{P}_m | n(p) = n\}.$$

It is trivial that $\mathcal{P}_m = \oplus_{n \geq 0} \mathcal{P}_m(n)$. By simple calculations, we get

Proposition 5.12 *(i) If m is odd (resp. even), then $\mathcal{P}_m(2n) = \emptyset$ (resp. $\mathcal{P}_m(2n-1) = \emptyset$).*

(ii) If $n < |m|$, then $\mathcal{P}_m(n) = \emptyset$.

We shall see the stability of $\mathcal{P}_m(n)$ by the actions of $\tilde{e}_i$ and $\tilde{f}_i$.

Proposition 5.13 *For a path $p \in \mathcal{P}_m(n)$, suppose that $\tilde{f}_i p \neq 0$ (resp. $\tilde{e}_i p \neq 0$), then we have $n(\tilde{f}_i p) = n(p)$ (resp. $n(\tilde{e}_i p) = n(p)$).*

Proof. For a path $p = (\cdots, i_k, i_{k+1} \cdots)$ and $i = 0, 1$, we set

$$a_k^{(i)} = \sum_{j<k} \varphi_i(i_j) - \varepsilon_i(i_{j+1}). \tag{5.19}$$

Remark. If $k \ll 0$ then $i_k = 0$, thus we have $a_k^{(i)} = 0$ for $k \ll 0$ and by the fact that $\varphi_i(\pm m) = \varepsilon_i(\mp m)$ we have $a_k^{(i)} = a_{k+1}^{(i)}$ for $k \gg 0$.

In order to prove the proposition, it is necessary to see the following lemma.

Lemma 5.14 *(i) For a path $p = (\cdots, i_k, i_{k+1} \cdots)$, if there exists $k \in \mathbf{Z}$ such that*

$$a_\nu^{(i)} \geq a_k^{(i)} \ (\nu < k) \text{ and } a_\nu^{(i)} > a_k^{(i)} \ (\nu > k), \tag{5.20}$$

then

$$\tilde{f}_i p = (\cdots, i_{k-1}, \tilde{f}_i(i_k), i_{k+1} \cdots), \tag{5.21}$$

otherwise $\tilde{f}_i p = 0$.

(ii) For a path $p = (\cdots, i_k, i_{k+1} \cdots) \in \mathcal{P}_m$, if there exists $k \in \mathbf{Z}$ such that

$$a_\nu^{(i)} > a_k^{(i)} \ (\nu < k) \text{ and } a_\nu^{(i)} \geq a_k^{(i)} \ (\nu > k), \tag{5.22}$$

then

$$\tilde{e}_i p = (\cdots, i_{k-1}, \tilde{e}_i(i_k), i_{k+1} \cdots), \tag{5.23}$$

otherwise $\tilde{e}_i p = 0$.

Proof of Lemma 5.14 Since the proof of (ii) is similar to the one of (i), we shall show only (i). For any $p = (\cdots, i_k, i_{k+1}, \cdots) \in \mathcal{P}_m(n)$ there exist $j, l \in \mathbf{Z}_{>0}$ such that $i_k = 0$ if $k \leq -j$ and $i_{2k} = m$ and $i_{2k+1} = -m$ if $k \geq l$. Then p is identified with

$$u_\infty \otimes i_{-j} \otimes i_{-j+1} \otimes \cdots \otimes i_{2l} \otimes i_{2l+1} \otimes t_\lambda \otimes u_{-\infty}. \tag{5.24}$$

For the vector (5.24), we set $A_{-j}^{(i)} := 0$ and

$$A_k^{(i)} = \sum_{-j \leq p < k} \varphi_i(i_p) - \varepsilon_i(i_{p+1}), \quad k = -j+1, \cdots, 2l. \tag{5.25}$$

Since $\varphi_i(u_\infty) = \varepsilon_i(u_\infty) = \varphi_i(i_{-j}) = \varepsilon_i(i_{-j}) = 0$, we get

$$A_k^{(i)} = a_k^{(i)} \quad \text{for } k = -j, \cdots, 2l. \tag{5.26}$$

By the previous remark, if there exists k which satisfies (5.20), $-j \le k \le 2l$. Therefore, by Proposition 2.1.1 (i) in [12] we obtain the desired result. $\square$

Now, let us show Proposition 5.13 (i). We shall consider $i = 1$ case. Suppose that for $p = (\cdots, i_{k-1}, i_k, i_{k+1} \cdots)$ we have $\tilde{f}_1\, p = (\cdots, i_{k-1}, \tilde{f}_1(i_k), i_{k+1} \cdots)$. We know that $\tilde{f}_1(i_k) = i_k + 1$. Thus, we get

$$\tilde{f}_1\, p = (\cdots, i_{k-1}, i_k + 1, i_{k+1} \cdots).$$

By Lemma 5.14, we have

$$a_{k-1}^{(1)} \ge a_k^{(1)} \text{ and } a_{k+1}^{(1)} > a_k^{(1)}.$$

By using this, we obtain,

$$0 \le a_{k-1}^{(1)} - a_k^{(1)} = -(\varphi_1(i_{k-1}) - \varepsilon_1(i_k)) = i_{k-1} + i_k, \tag{5.27}$$

$$0 < a_{k+1}^{(1)} - a_k^{(1)} = \varphi_1(i_k) - \varepsilon_1(i_{k+1}) = -i_k - i_{k+1}. \tag{5.28}$$

By (5.27) and (5.28), we get

$$i_{k-1} + i_k \ge 0 \quad \text{and} \quad i_k + i_{k+1} < 0.$$

Therefore,

$$\begin{aligned} |i_{k-1} + \tilde{f}_1(i_k)| &= |i_{k-1} + i_k + 1| = |i_{k-1} + i_k| + 1, \\ |\tilde{f}_1(i_k) + i_{k+1}| &= |i_k + i_{k+1} + 1| = |i_k + i_{k+1}| - 1. \end{aligned}$$

Then $n(\tilde{f}_1\, p) = n(p)$. By arguing similarly we can prove for $i = 0$. Now, we have completed the proof of Proposition 5.13. $\square$

6 Path-Spin Correspondence

The purpose of this section is to give a strict morphism of P_{cl}-weighted crystals $\mathcal{P}_m(n) \longrightarrow B^{\otimes n}$. (*cf.* 2.2)

Now, we shall define a map from $\mathcal{P}_m(n)$ to $B^{\otimes n}$ as follows: For $p \in \mathcal{P}_m(n)$, let $(\iota_1, \iota_2, \cdots, \iota_n)$ be the sequence of wall types (ordered from the left to the right). The map $\psi : \mathcal{P}_m(n) \longrightarrow B^{\otimes n}$ is given by

$$\psi(p) = (-\iota_1) \otimes (-\iota_2) \otimes \cdots \otimes (-\iota_n), \tag{6.1}$$

for any $p \in \mathcal{P}_m(n)$.

Theorem 6.1 *The map ψ is a strict morphism of P_{cl}-weighted crystals from $\mathcal{P}_m(n)$ to $B^{\otimes n}$.*

Proof. In order to prove the theorem, we shall see that ψ satisfies

$$\begin{aligned} wt(p) &= wt(\psi(p)), && (6.2)\\ \varepsilon_i(p) &= \varepsilon_i(\psi(p)),\ \varphi_i(p) = \varphi_i(\psi(p)), && (6.3)\\ \tilde{f}_i\psi(p) &= \psi(\tilde{f}_i\, p), && (6.4)\\ \tilde{e}_i\psi(p) &= \psi(\tilde{e}_i\, p), && (6.5) \end{aligned}$$

for any $p \in \mathcal{P}_m(n)$ and $i = 0,\ 1$.

A m-path $g = (g_k)_{k\in\mathbf{Z}}$ satisfying $g_k = 0$ for $k < 0$, $g_{2k} = m$ and $g_{2k+1} = -m$ for $k \geq 0$ is called *m-ground-state path*. $g = (g_k)_{k\in\mathbf{Z}}$ just corresponds to $u_\infty \otimes t_\lambda \otimes u_{-\infty}$ in $B(\infty)\otimes T_\lambda \otimes B(-\infty)$. Then $wt(g) = m(\Lambda_0 - \Lambda_1)$. Therefore, for $p = (i_k)_{k\in\mathbf{Z}}$ the following formula is obtained easily;

$$\begin{aligned} wt(p) &= m(\Lambda_0 - \Lambda_1) + \textstyle\sum_{k\in\mathbf{Z}}(wt(i_k) - wt(g_k)) \\ &= (m + 2\textstyle\sum_{k\in\mathbf{Z}}(i_k - g_k))(\Lambda_0 - \Lambda_1). \end{aligned} \qquad (6.6)$$

By the definition of path, we know that the summation in (6.6) is finite. Therefore, by the fact $g_{k-1} + g_k = 0$ ($k \neq 0$) and $g_{-1} + g_0 = m$,

$$\begin{aligned} wt(p) &= (m + \sum_{k\in\mathbf{Z}}(i_{k-1} + i_k - g_{k-1} - g_k)(\Lambda_0 - \Lambda_1) \\ &= (\sharp\{(+)\text{walls}\} - \sharp\{(-)\text{walls}\})(\Lambda_0 - \Lambda_1) \\ &= wt(\psi(p)). \end{aligned}$$

Here note that $wt(\pm) = \pm(\Lambda_1 - \Lambda_0)$. Now we get (6.2).

Let us show (6.3). For $p = (\cdots, i_k, i_{k+1}, \cdots) \in \mathcal{P}_m(n)$, let a and b be sufficiently large integers such that $i_{-j} = 0$, $i_{2k} = m$ and $i_{2k+1} = -m$ for any $j > a$ and $k > b$. Therefore, since p is identified with $u_\infty \otimes i_{-j} \otimes \cdots \otimes i_{2k} \otimes i_{2k+1} \otimes t_\lambda \otimes u_{-\infty}$ and $\varphi_i(u_{-\infty}) = \varepsilon_i(u_{-\infty}) = 0$, by (2.20) we have

$$\varphi_i(p) = \varphi_i(u_\infty \otimes i_{-j} \otimes \cdots \otimes i_{2k} \otimes i_{2k+1} \otimes t_\lambda), \qquad (6.7)$$

for $j > a$ and $k > b$. By the formula $\varphi_i(t_\lambda) = -\infty$, Proposition 2.1.1 (0) in [12] and (2.20), we get

$$\varphi_i(p) = \langle h_i, \lambda\rangle + \varphi_i(i_{2k+1}) + \max_{-j\leq p\leq 2k+1}(a^{(i)}_{2k+1} - a^{(i)}_p). \qquad (6.8)$$

We shall consider $i = 1$ case. Then (6.8) can be written explicitly as follows:

$$\varphi_1(p) = \max_{-j\leq p\leq 2k+1}(-\sum_{p<s\leq 2k+1} i_{s-1} + i_s), \qquad (6.9)$$

by using $\varphi_1(i_{2k+1}) = -i_{2k+1} = m = -\langle h_1, \lambda\rangle$.

Let $k_1, k_2, \cdots, k_s$ ($s \le n$) be the sequence of positions of walls in p such that $k_j < k_{j+1}$ and there is no wall in $j \ne k_1, \cdots, k_s$. Here note that since more than one walls can occupy the same position, $s \le n$. Let c_i be the position of i-th wall. For $j = 1, 2, \cdots\ s$, we set

$$N_j^{\pm} := \sharp\left\{\iota_r = \pm \,|\, c_r \in \{k_j, \cdots, k_s\}\right\}.$$

Since $i_{c-1} + i_c = 0$ if $c \not\subset \{k_1, \cdots, k_r\}$, The formula (6.9) can be written as follows;

$$\begin{aligned} \varphi_1(p) &= {\max}^*{}_{1\le j\le s}\left\{-\textstyle\sum_{l=j}^{s} i_{k_l-1} + i_{k_l}\right\}, \\ &= {\max}^*{}_{1\le j\le s}\{N_j^- - N_j^+\}, \end{aligned} \tag{6.10}$$

where $\max^*\{z_1, \cdots, z_n\} := \max\{z_1, \cdots, z_n, 0\} \ge 0$. Note that if there is no $(-)$ wall in p, $\varphi_1(p) = 0$ and $\varphi_1(\psi(p)) = \varphi_1((-)^{\otimes n}) = 0$. Then we may assume that there exists $(-)$ wall in p.

We shall investigate $\varphi_1(\psi(p))$. By Proposition 2.1.1 (0) in [12], we can get the following,

$$\begin{aligned} \varphi_1(\psi(p)) &= \varphi_1((-\iota_1) \otimes \cdots \otimes (-\iota_n)) \\ &= \max_{1\le j\le n}\left\{\sum_{j\le k\le n} \varphi_1(-\iota_k) - \sum_{j<k\le n} \varepsilon_1(-\iota_k)\right\}. \end{aligned} \tag{6.11}$$

Here note that $\varphi_1(+) = 1 = \varepsilon_1(-)$ and $\varphi_1(-) = 0 = \varepsilon_1(+)$ by (4.3), $\sum_{j\le k\le n}\varphi_1(-\iota_k) = \sharp\{\iota_k = -\,;\, j \le k \le n\}$ and $\sum_{j<k\le n}\varepsilon_1(-\iota_k) = \sharp\{\iota_k = +\,;\, j < k \le n\}$. Then we can rewrite (6.11) as follows;

$$\varphi_i(\psi(p)) = \max_{1\le j\le n}\{\sharp\{\iota_k = -\,;\, j \le k \le n\} - \sharp\{\iota_k = +\,;\, j < k \le n\}\}. \tag{6.12}$$

Therefore, if t ($1 \le t \le n$) gives the maximum in (6.12), there are two cases

(i) $\iota_t = -$ and $\iota_{t-1} = +$ ($t > 1$).

(ii) $t = 1$ and $\iota_1 = -$.

Since in both cases $\varepsilon_1(-\iota_t) = \varepsilon_1(+) = 0$ and $\sum_{j\le k\le n}\varepsilon_1(-\iota_k) \ge \sum_{j<k\le n}\varepsilon_1(-\iota_k)$, we can rewrite (6.12) to

$$\varphi_i(\psi(p)) = \max_{1\le j\le n}\{\sharp\{\iota_k = -\,;\, j \le k \le n\} - \sharp\{\iota_k = +\,;\, j \le k \le n\}\}. \tag{6.13}$$

Since we have the following by the definition of $N_j^{\pm}$,

$$\left\{N_j^- - N_j^+\right\}_{1\le j\le s} \subset \{\sharp\{\iota_k = -\,;\, j \le k \le n\} - \sharp\{\iota_k = +\,;\, j \le k \le n\}\}_{1\le j\le n},$$

by (6.10) and (6.13) we get $\varphi_1(p) \leq \varphi_i(\psi(p))$. We set

$$S := \{s \mid \begin{array}{l} 1 \leq s \leq n,\ s\text{-th wall in } p \text{ is a } (-) \text{ wall and} \\ \text{the left-most wall among walls in the same position} \end{array} \}. \tag{6.14}$$

The cases (i) and (ii) as above mean that if t gives the maximum of (6.13), $t \in S$. Here note that if $s \in S$,

$$N_s^{\pm} = \sharp\{\iota_k = \pm; s \leq k \leq n\}.$$

Therefore, we get $\varphi_1(p) \geq \varphi_1(\psi(p))$. Now, we have $\varphi_1(p) = \varphi_1(\psi(p))$. As for φ_0-case and ε_i-case arguing similarly, we obtain (6.3).

Let us show (6.4) for $i = 1$. For $p = (\cdots, i_{j-1}, i_j, i_{j+1} \cdots)$ we assume that there exists k satisfying (5.20) for $i = 1$, *i.e.* $\tilde{f}_1 p = (\cdots, i_{k-1}, \tilde{f}_1(i_k), i_{k+1}, \cdots)$. We know that $a_k^{(1)}$ is given by $a_k^{(1)} = -\sum_{j<k} i_j + i_{j+1}$. Since k satisfies (5.20) for $i = 1$, we have $a_k^{(1)} < a_{k+1}^{(1)}$ and $a_{k-1}^{(1)} \geq a_k^{(1)}$. Then we get

$$i_k + i_{k+1} = a_k^{(1)} - a_{k+1}^{(1)} < 0, \tag{6.15}$$

$$i_{k-1} + i_k = a_{k-1}^{(1)} - a_k^{(1)} \geq 0. \tag{6.16}$$

By (6.15), we know that there exist $(-)$ walls in position $k+1$. By this and the fact

$$\tilde{f}_1(i_k) = i_k + 1, \tag{6.17}$$

we get

$$|\tilde{f}_1(i_k) + i_{k+1}| = |i_k + i_{k+1}| - 1. \tag{6.18}$$

By (6.16) and (6.17),

$$|i_{k-1} + \tilde{f}_1(i_k)| = |i_{k-1} + i_k| + 1, \tag{6.19}$$

Let j-th wall in p be the left-most wall among walls in position $k+1$ (the existence is guaranteed by (6.15)). (6.15), (6.18) and (6.19) imply that the j-th wall and other walls in position $k+1$ are $(-)$ walls and the j-th wall is changed by the action of $\tilde{f}_1$ to $(+)$ wall in position k. Then j belongs to S.

That is, let $(\iota_1, \iota_2, \cdots, \iota_n)$ and $(\iota'_1, \iota'_2, \cdots, \iota'_n)$ be the sequences of wall types of p and $\tilde{f}_1 p$ respectively, we have

$$(\iota_1, \cdots, \overset{j}{-}, \cdots, \iota_n) \xrightarrow{\tilde{f}_1} (\iota_1, \cdots, \overset{j}{+}, \cdots, \iota_n) = (\iota'_1, \cdots, \iota'_n). \tag{6.20}$$

By (6.20), we know that

$$\psi(p) = (-\iota_1) \otimes \cdots \otimes \overset{j}{+} \otimes \cdots \otimes (-\iota_n),\ \ \psi(\tilde{f}_1 p) = (-\iota_1) \otimes \cdots \otimes \overset{j}{-} \otimes \cdots \otimes (-\iota_n). \tag{6.21}$$

By (6.21), it is sufficient to show the following,

$$\tilde{f}_1\left((-\iota_1)\otimes\cdots\otimes\overset{j}{+}\otimes\cdots\otimes(-\iota_n)\right)=(-\iota_1)\otimes\cdots\otimes\overset{j}{-}\otimes\cdots\otimes(-\iota_n). \quad (6.22)$$

For p with $\psi(p)=(-\iota_1)\otimes\cdots\otimes(-\iota_n)$ we shall define the function $\overline{a}_k$ as follows; (this just coincides with a_k in Proposition 2.1.1 (0) in [12] up to the first term.).

$$\overline{a}_k := -\varepsilon_1(-\iota_1)+\sum_{1\le l<k}\varphi_1(-\iota_l)-\varepsilon_1(-\iota_{l+1}). \quad (6.23)$$

It is easy to translate (6.23) to the following form by (4.3);

$$\overline{a}_k=\sharp\{\iota_l=-\,|\,1\le l<k\}-\sharp\{\iota_l=+\,|\,1\le l\le k\}. \quad (6.24)$$

By Proposition 2.1.1 (i) in [12], we know that if there exists j satisfying

$$\overline{a}_\nu\ge\overline{a}_j \ \text{ for } \ \nu<j \ \text{ and } \overline{a}_\nu>\overline{a}_j \ \text{ for } \ j<\nu, \quad (6.25)$$

we have

$$\tilde{f}_1\left((-\iota_1)\otimes\cdots\otimes(-\iota_j)\otimes\cdots\otimes(-\iota_n)\right)=(-\iota_1)\otimes\cdots\otimes\tilde{f}_1(-\iota_j)\otimes\cdots\otimes(-\iota_n). \quad (6.26)$$

Then we shall show that j as in (6.20) and (6.21) satisfies (6.25). Since the position of the j-th wall is $k+1$ and there is no $(+)$ wall in position $k+1$ by the argument above, we get

$$\sharp\{\iota_l=-\,|\,1\le l<j\} = \sum_{\substack{r\le k\\ i_{r-1}+i_r<0}}|i_{r-1}+i_r| = -\sum_{\substack{r\le k\\ i_{r-1}+i_r<0}}i_{r-1}+i_r, \quad (6.27)$$

$$\sharp\{\iota_l=+\,|\,1\le l\le j\} = \sum_{\substack{r\le k+1\\ i_{r-1}+i_r>0}}|i_{r-1}+i_r| = \sum_{\substack{r\le k\\ i_{r-1}+i_r>0}}i_{r-1}+i_r. \quad (6.28)$$

The following is obtained by (6.24), (6.27) and (6.28),

$$\overline{a}_j=-\sum_{r\le k}i_{r-1}+i_r=\sum_{r<k}\varphi_1(i_r)-\varepsilon_1(i_{r+1})=a_k^{(1)}, \quad (6.29)$$

By the form of (6.24), we know that j satisfying (6.25) belongs to S. Therefore, in order to show that j satisfies (6.25) it is enough to show that j satisfies

$$\overline{a}_\nu\ge\overline{a}_j \ \text{ for } \ \nu<j \ (j,\nu\in S) \ \text{ and } \overline{a}_\nu>\overline{a}_j \ \text{ for } \ j<\nu, \ (j,\nu\in S). \quad (6.30)$$

By the same argument as for obtaining (6.29), we can see that for any $\nu<j$ (resp. $\nu>j$) $(\nu,j\in S)$ there exists t such that

$$t<k \text{ (resp. } t>k) \ \text{ and } \ \overline{a}_\nu=a_t^{(1)}. \quad (6.31)$$

By (5.20) for $i = 1$, (6.29) and (6.31), we get that j satisfies (6.30) and then (6.25). Now, we get (6.26).

Next, we shall show that if $\tilde{f}_1 p = 0$, $\tilde{f}_1 \psi(p) = 0$. We assume that $\tilde{f}_1 p = 0$ and set $\xi = \max\{\nu \mid i_{\nu-1} + i_\nu \neq 0\}$. By Lemma 5.14 we know that ξ satisfies

$$a_\nu^{(1)} \geq a_\xi^{(1)} \quad \text{for } \nu > \xi \text{ and } a_\xi^{(1)} = a_\nu^{(1)} \text{ for } \xi > \nu. \tag{6.32}$$

Now, we set $F := a_\xi^{(1)}$. Let us assume that $a_{\xi-1}^{(1)} = a_\xi^{(1)}$. Then we have

$$0 = a_{\xi-1}^{(1)} - a_\xi^{(1)} = -\varphi_1(i_{\xi-1}) + \varepsilon_1(i_\xi) = i_{\xi-1} + i_\xi.$$

This contradicts the definition of ξ. Thus, we get $a_{\xi-1}^{(1)} > a_\xi^{(1)}$ and then $i_{\xi-1} + i_\xi > 0$. Furthermore, by the fact that $i_{\xi-1} + i_\xi > 0$ we have

$$\iota_n = +. \tag{6.33}$$

Here note that

$$a_\xi^{(1)} = F = \overline{a}_n. \tag{6.34}$$

It is sufficient to show that

$$\tilde{f}_1 \psi(p) = \tilde{f}_1 \left((-\iota_1) \otimes \cdots \otimes (-\iota_n)\right) = (-\iota_n) \otimes \cdots \otimes \tilde{f}_1(-\iota_n), \tag{6.35}$$

since we know that (6.33) and $\tilde{f}_1(-) = 0$. Since we know that if there is j with (6.25), (6.26) holds. in order to show (6.35), we shall prove

$$\overline{a}_\nu \geq \overline{a}_n, \quad \text{for } \nu < n. \tag{6.36}$$

We assume that there exists j such that $j \neq n$ and satisfies (6.25). Let t be the position of the j-th wall. It is easy to see that $\iota_j = -$ by (6.24). Thus, by (6.33), we have

$$t < \xi. \tag{6.37}$$

By simliar argument to the one for obtaining (6.31), we get $\overline{a}_j = a_t^{(1)}$. Therefore, by (6.32) and (6.34) we have

$$\overline{a}_j \geq F = \overline{a}_n. \tag{6.38}$$

This contradicts the definition of j satisfying (6.25). Now we get (6.35) and then $\tilde{f}_1 \psi(p) = 0$ if $\tilde{f}_1 p = 0$.

By arguing similarly, we obtain $\tilde{f}_0 \psi(p) = \psi(\tilde{f}_0 p)$ and $\tilde{e}_i \psi(p) = \psi(\tilde{e}_i p)$. Then, we have completed the proof of Theorem 6.1. $\square$

7 Classification of Path

In this section, we shall describe every connected component in $B(\widetilde{U}_q(\mathfrak{g}))$. The notations in this section follow the previous sections.

7.1 Domain type and Domain parameter

For a path $p \in \mathcal{P}_m(n)$ $(n \geq 0,\ m \in \mathbf{Z})$, let $d_0, d_1, \cdots, d_{n-1}, d_n$ be the sequence of domains in p. The domains d_0 and d_n are infinite domains. Here note that as stated in Remark (iii) in 5.3,

$$\text{every domain with non-zero length is in the form:} \quad \cdots, k, -k, k, -k, \cdots, \tag{7.1}$$

where k is an integer.

Definition 7.1 *For a domain d_j with non-zero length, fixing some entry i_ν in d_j and its position ν, the domain type $t(d_j)$ of d_j is given by*

$$t(d_j) := (-1)^\nu i_\nu. \tag{7.2}$$

Remark.

(i) By (7.1), this definition is well-defined, *i.e.*, a domain type is uniquely determined.

(ii) Domain type of domain d_0 is always 0 and one of domain d_n is always m by the definition of $\mathcal{P}_m(n)$.

The following lemma is trivial.

Lemma 7.2 *For a path p let i_{k-1} and i_k be entries in p with $|i_{k-1} + i_k| \neq 0$ and let d_j and d_l $(j < l)$ be domains including i_{k-1} and i_k respectively. Then we have*

$|t(d_l) - t(d_j)| - 1 =$ *the number of domains with zero-length between d_j and d_l.*

By this lemma, the following definition is well-defined.

Definition 7.3 *Let d_r be the i-th zero-length domain between d_j and d_l as in Lemma 7.2. Domain type $t(d_r)$ is given by: $t(d_r) = t(d_j) + i$ if $t(d_j) < t(d_l)$ and $t(d_r) = t(d_j) - i$ if $t(d_j) > t(d_l)$.*

Example 7.4 *For* $p = (\overset{\text{position}}{\cdots\cdots}, \overset{-2}{0}, \overset{-1}{0}, \overset{0}{2}, \overset{1}{-1}, \overset{2}{3}, \overset{3}{-3}, \overset{4}{3}, \cdots)$, *we shall visualize walls and domains as follows:*

$$\cdots \overset{d_0}{0}\overset{d_1}{0|}\ \overset{d_2}{|2|} \overset{d_3}{-1|}\ \overset{d_4}{|}3 \overset{d_5}{-}33 \cdots.$$

We know that there are five walls and four finite domains in p. Let d_1, d_2, d_3 and d_4 be the four finite domains. The domains d_1 and d_4 are zero-length domains. The domain type of these four domains are $1, 2, 1, 2$ respectively. Of course, the domain type of the leftmost infinite domain is 0 and the one of the rightmost infinite domain is 3.

Remark. Note that for any path $p \in \mathcal{P}_m(n)$ and $j = 0, 1, \cdots, n-1$

$$|t(d_{j+1}) - t(d_j)| = 1. \tag{7.3}$$

Definition 7.5 *A sequence of integers $t_1, t_2, \cdots, t_{n-1}$ is in m-domain configuration if $|t_j - t_{j-1}| = 1$ for $j = 1, \cdots, n$, where $m \in \mathbf{Z}$, $t_0 = 0$ and $t_n = m$.*

The following lemma is trivial.

Lemma 7.6 *There exists a sequence $t_1, \cdots, t_{n-1}$ in m-domain configuration if and only if $n - |m| \in 2\mathbf{Z}_{\geq 0}$.*

By the above remark, we get

Lemma 7.7 *A sequence of domain types for any path in $\mathcal{P}_m$ is in m-domain configuration.*

Definition 7.8 (i) *Let $\vec{t} = (t_1, t_2, \cdots, t_{n-1})$ be in a m-domain configuration,*

- *(a) t_j is regular in $\vec{t}$ if $t_{j-1} - t_j = t_j - t_{j+1}$.*
- *(b) t_j is up (resp. down)-regular in $\vec{t}$ if t_j is regular in $\vec{t}$ and $t_{j-1} < t_j < t_{j+1}$ (resp. $t_{j-1} > t_j > t_{j+1}$).*
- *(c) t_j is critical in $\vec{t}$ if $t_{j-1} - t_j = -t_j + t_{j+1}$.*
- *(d) t_j is maximal (resp. minimal) in $\vec{t}$ if t_j is critical in $\vec{t}$ and $t_{j-1} + 1 = t_j = t_{j+1} + 1$ (resp. $t_{j-1} - 1 = t_j = t_{j+1} - 1$).*

Here $t_0 = 0$ and $t_n = m$.

(ii) *For a path $p \in \mathcal{P}_m(n)$, let $d_1, \cdots, d_{n-1}$ be its finite domains and $\vec{t}(\vec{d}) = (t(d_1), \cdots, t(d_{n-1}))$ be the sequence of their domain types.*

- *(a) d_j is a regular domain if $t(d_j)$ is regular in $\vec{t}(\vec{d})$.*
- *(b) d_j is* up-regular *(resp.* down-regular*) if $t(d_j)$ is up-regular in $\vec{t}(\vec{d})$, in particular, d_0 is* up *(resp.* down*) if $t(d_0) < t(d_1)$ (resp. $t(d_0) > t(d_1)$ and d_n is* up *(resp.* down*) if $t(d_{n-1}) < t(d_n)$ (resp. $t(d_{n-1}) > t(d_n)$).*
- *(c) d_j is a critical domain if $t(d_j)$ is critical in $\vec{t}(\vec{d})$.*
- *(d) d_j is* maximal *(resp.* minimal*) if $t(d_j)$ is maximal (resp. minimal) in $\vec{t}(\vec{d})$.*

Remark.

(i) By Definition 7.3, any zero-length domain is a regular domain.

(*ii*) If $t_1 \cdots, t_{n-1}$ is in m-domain configuration, any t_j ($j = 1, \cdots, n-1$) is classified in Definition 7.8 (i) (b)(d) and then any domain is classified in Definition 7.8 (ii) (b)(d).

Example 7.9 *In Example 7.4, the infinite domains d_0 and d_5 are up. d_1 and d_4 are up-regular, d_2 is maximal and d_3 is minimal.*

Definition 7.10 *For $p \in \mathcal{P}_m(n)$, let $d_1, d_2, \cdots, d_{n-1}$ be its finite domains and $l(d_1), l(d_2), \cdots, l(d_{n-1})$ be their lengths. Domain parameter $c(d_j)$ is given by*

$$\textit{if } d_j \textit{ is a regular domain, } c(d_j) := [[\frac{l(d_j)}{2}]],$$
$$\textit{if } d_j \textit{ is a critical domain, } c(d_j) := [[\frac{l(d_j)+1}{2}]],$$

where $[[n]]$ =the maximum integer which is less than or equal to n.

Let $\vec{t} = (t_1, t_2, \cdots, t_{n-1})$ be in a m-domain configuration and $\vec{c} = (c_1, c_2, \cdots, c_{n-1})$ be a sequence of non-negative integers. For $\vec{t}$ and $\vec{c}$, we set

$$\mathcal{P}_m(n; \vec{t}; \vec{c}) := \left\{ p \in \mathcal{P}_m(n) \mid \begin{array}{l} t(d_j) = t_j \text{ and } c(d_j) = c_j \text{ for any } j = 1, 2, \cdots, n-1, \\ \text{where } d_1, \cdots, d_{n-1} \text{ are domains in } p \end{array} \right\}.$$

The following proposition guarantees the existence of $\mathcal{P}_m(n; \vec{t}; \vec{c})$.

Proposition 7.11 *Suppose that $n - |m| \in 2\mathbf{Z}_{\geq 0}$. For any $\vec{t} = (t_1, \cdots, t_{n-1})$ in m-domain configuration and any sequence of non-negative integers $\vec{c} = (c_1, \cdots, c_{n-1})$,*

$$\mathcal{P}_m(n; \vec{t}; \vec{c}) \neq \emptyset \tag{7.4}$$

Proof. By Lemma 7.6, if $n - |m| \in 2\mathbf{Z}_{\geq 0}$, there exists $\vec{t} = (t_1, \cdots, t_{n-1})$ in m-domain configuration. Let $p_l^{(\pm)}$ be paths given as follows: for $j = 1, \cdots, n-1$

$$d_j := \begin{cases} \underbrace{\pm t_j, \mp t_j, \cdots, \pm t_j, \mp t_j}_{2c_j} & \text{if } t_j \text{ is up-regular,} \\ \underbrace{\mp t_j, \pm t_j, \cdots, \mp t_j, \pm t_j}_{2c_j} & \text{if } t_j \text{ is down-regular,} \\ \underbrace{\pm t_j, \mp t_j, \cdots, \mp t_j, \pm t_j}_{2c_j+1} & \text{if } t_j \text{ is maximal,} \\ \underbrace{\mp t_j, \pm t_j, \cdots, \pm t_j, \mp t_j}_{2c_j+1} & \text{if } t_j \text{ is minimal,} \end{cases} \tag{7.5}$$

$$d_n := \begin{cases} \pm m, \mp m, \cdots & \text{if } d_n \text{ is up}, \\ \mp m, \pm m, \cdots & \text{if } d_n \text{ is down}. \end{cases} \tag{7.6}$$

Now, we order these domains by setting the position of the left most m (resp. $-m$) in d_n being $2l$ (resp. $2l-1$). For example,

$$p_l^{(+)} = (\cdots 00|d_1|d_2|\cdots|d_{n-1}|\overset{2l}{m} -m \cdots) \ \text{ or } \ (\cdots 00|d_1|d_2|\cdots|d_{n-1}|-m \overset{2l}{m} \cdots).$$

For $p_l^{(+)}$, by using induction on the index of domains we shall show the claim that the position of any entry t_j in d_j is even and the one of $-t_j$ in d_j is odd. Now we assume that d_n is up. Then d_{n-1} must be up-regular or minimal by Definition 7.8. It is trivial that in both cases by (7.5) the position of t_{n-1} is even and the one of $-t_{n-1}$ is odd. Now, we assume that for $i = j+1$ the claim is valid. If t_{j+1} is up-regular or maximal, by Definition 7.8, t_j must be up-regular or minimal. Then by (7.5) we have

$$(\cdots d_j|d_{j+1}\cdots) = (\cdots t_j, -t_j|t_{j+1}, -t_{j+1}, \cdots). \tag{7.7}$$

This implies that the statement is valid for $i = j$. If t_{j+1} is down-regular or minimal, by Definition 7.8, t_j must be down-regular or maximal. Then by (7.5) we have

$$(\cdots d_j|d_{j+1}\cdots) = (\cdots - t_j, t_j| - t_{j+1}, t_{j+1}, \cdots). \tag{7.8}$$

This implies that the statement is valid for $i = j$. Therefore, we have

$$t_j = t(d_j) \ \text{ and then } \ c_j = c(d_j).$$

We obtain that $p_l^{(+)} \in \mathcal{P}_m(n;\vec{t};\vec{c})$. We can also show for $p_l^{(-)}$. □

Reamrk. These $p_l^{(\pm)}$ are extremal vectors in $\mathcal{P}_m(n;\vec{t};\vec{c})$, which is shown in 7.3. We also know that all walls in $p_l^{(+)}$ (resp. $p_l^{(-)}$) are + (resp. $-$) simultaneously by the definition of $p_l^{(\pm)}$ and the conditions in Definition 7.8. In fact, in (7.7) (resp. (7.8)), we get $t_{j+1} - t_j = 1$ and (resp. $t_j - t_{j+1} = 1$).

7.2 Stability of $\mathcal{P}_m(n;\vec{t},\vec{c})$

We shall show the stability of $\mathcal{P}_m(n;\vec{t};\vec{c})$ by the actions of $\tilde{e}_i$ and $\tilde{f}_i$.

Proposition 7.12 *For any $i \in I$, we have*

$$\begin{aligned} \tilde{e}_i\mathcal{P}_m(n;\vec{t};\vec{c}) &\subset \mathcal{P}_m(n;\vec{t};\vec{c}) \sqcup \{0\}, \\ \tilde{f}_i\mathcal{P}_m(n;\vec{t};\vec{c}) &\subset \mathcal{P}_m(n;\vec{t};\vec{c}) \sqcup \{0\}. \end{aligned} \tag{7.9}$$

In order to show this proposition, we shall prepare several lemmas.

Lemma 7.13 *For $p = (\cdots, i_{k-1}, i_k, i_{k+1}, \cdots) \in \mathcal{P}_m(n)$, suppose that $\tilde{f}_i p = (\cdots, i_{k-1}, \tilde{f}_i(i_k), i_{k+1}, \cdots)$ (resp. $\tilde{e}_i p = (\cdots, i_{k-1}, \tilde{e}_i(i_k), i_{k+1}, \cdots)$) and let d_j be the domain including i_k. Then we have*

(i) *The entry i_k is the right-most entry (resp. left-most entry) in d_j.*

(ii) *Suppose that d_j is a finite domain. The length $l(d_j)$ is odd if and only if d_j is regular and the length $l(d_j)$ is even if and only if d_j is critical.*

(iii) *Suppose that d_{j+1} is a finite domain. The length $l(d_{j+1})$ (resp. $l(d_{j-1})$) is even if and only if d_{j+1} (resp. d_{j-1}) is regular and the length $l(d_{j+1})$ (resp. $l(d_{j-1})$) is odd if and only if d_{j+1} (resp. d_{j-1}) is critical.*

Remark. The statement (i) means that $d_j \neq d_n$ (resp. $d_j \neq d_0$), that is , there is a domain on the right (resp. left) side of d_j. Then, the statement (iii) makes sense.

Proof. Since the proof for the $\tilde{e}_i$ case is similar to the one for $\tilde{f}_i$, we shall show only for the $\tilde{f}_i$ case.

(i) By Lemma 5.14 (i), the hypothesis $\tilde{f}_i p = (\cdots, i_{k-1}, \tilde{f}_i(i_k), i_{k+1}, \cdots)$ implies that $a_k^{(i)} < a_{k+1}^{(i)}$ and then we have

$$i_k + i_{k+1} < 0 \quad \text{if } i = 1 \ , \tag{7.10}$$

$$i_k + i_{k+1} > 0 \quad \text{if } i = 0. \tag{7.11}$$

Then we get $|i_k + i_{k+1}| > 0$. This gives the desired result.

(ii) We shall show the $\tilde{f}_1$-case. Let i_r be the left-most entry in d_j. (by (i) the right-most entry is i_k, then $r \leq k$.). We set $t := t(d_j)$, then, $i_r = \pm t$ and $i_k = \pm t$. Let us recall $a_k^{(i)}$ in (5.19). Owing to (7.1) and $\varphi_i(x) - \varepsilon_i(-x) = 0$, we have $a_r^{(1)} = a_k^{(1)}$. Then by Lemma 5.14, we get $a_r^{(1)} = a_k^{(1)} \leq a_{r-1}^{(1)}$ and then

$$0 \leq a_{r-1}^{(1)} - a_r^{(1)} = -\varphi_1(i_{r-1}) + \varepsilon_1(i_r) = i_{r-1} + i_r. \tag{7.12}$$

The definition of i_r that i_r is the left-most entry in d_j implies that there are walls in position r and then $i_{r-1} + i_r \neq 0$. Thus, due to (7.12) we get

$$i_{r-1} + i_r > 0. \tag{7.13}$$

There are the following cases (a)–(d):

(a) $i_r = i_k = t$. (*i.e.* r and k are even.)

(b) $i_r = i_k = -t$. (*i.e.* r and k are odd.)

(c) $i_r = -t$ and $i_k = t$. (*i.e.* r is odd and k is even.)

(d) $i_r = t$ and $i_k = -t$. (*i.e.* r is even and k is odd.)

In fact, the condition (a) or (b) is equivalent to that $l(d_j) = k - r + 1$ is odd and the condition (c) or (d) is equivalent to that $l(d_j) = k - r + 1$ is even. Since these (a)–(d) covers all possibilities for d_j, it is enough to show that if (a) or (b), d_j is regular and if (c) or (d), d_j is critical. Let d_s and d_p be the domains including i_{r-1} and i_{k+1} respectively.

In the case (a) (resp. (b)), by (7.10) and (7.13), we get $i_{k+1} < -t$ (resp. $i_{k+1} < t$) and $i_{r-1} > -t$ (resp. $i_{r-1} > t$). Since $k + 1$ and $r - 1$ are odd (resp. even), the domain types $t(d_p) = -i_{k+1} > t$ (resp. $t(d_p) = i_{k+1} < t$) and $t(d_s) = -i_{r-1} < t$ (resp. $t(d_s) = i_{r-1} > t$). This implies

$$t(d_{j+1}) = t + 1,\ t(d_{j-1}) = t - 1 \quad (\text{resp. } t(d_{j+1}) = t - 1,\ t(d_{j-1}) = t + 1.). \tag{7.14}$$

Furthermore, this (7.14) implies that the domain d_j is regular.

In the case (c) (resp. (d)), by (7.10) and (7.13), we get $i_{k+1} < -t$ (resp. $i_{k+1} < t$) and $i_{r-1} > t$ (resp. $i_{r-1} > -t$). Since $k + 1$ is odd (resp. even) and $r - 1$ is even (resp. odd), the domain types $t(d_p) = -i_{k+1} > t$ (resp. $t(d_p) = i_{k+1} < t$) and $t(d_s) = i_{r-1} > t$ (resp. $t(d_s) = -i_{r-1} < t$). This implies that

$$t(d_{j+1}) = t + 1,\ t(d_{j-1}) = t + 1 \quad (\text{resp. } t(d_{j+1}) = t - 1,\ t(d_{j-1}) = t - 1.). \tag{7.15}$$

Furthermore, this (7.15) means that the domain d_j is critical. Now, we have completed the proof of (ii)

(iii) We shall show the $\tilde{f}_1$-case. Since $i_k + i_{k+1} < 0$ by (7.10), we shall consider the following two cases:

(1) $i_k + i_{k+1} \leq -2$

(2) $i_k + i_{k+1} = -1$.

(1) The assumption $i_k + i_{k+1} \leq -2$ implies that the domain d_{j+1} is a domain with zero-length. By Remark under Definition 7.8, d_{j+1} is a regular domain.

(2) The assumption $i_k + i_{k+1} = -1$ means that $i_{k+1} = \pm t - 1$ ($t = t(d_j)$) and there is only one wall in position $k + 1$. Then we know that i_{k+1} is included in the domain d_{j+1} and i_{k+1} is the left-most entry of d_{j+1}. Let i_l be the right-most entry of d_{j+1} ($k + 1 \leq l$).

By the definition of $a_k^{(i)}$, we have

$$a_{k+1}^{(1)} = a_k^{(1)} + \varphi_1(i_k) - \varepsilon_1(i_{k+1}) = a_k^{(1)} - (i_k + i_{k+1}) = a_k^{(1)} + 1. \tag{7.16}$$

By Lemma 5.14 if $\nu > k$, then $a_\nu^{(1)} > a_k^{(1)}$. Then by (7.16) we have

$$a_\nu^{(1)} \geq a_{k+1}^{(1)} \ (\nu \geq k + 1). \tag{7.17}$$

Owing to (7.1), we can easily get

$$a_{l+1}^{(1)} = a_{k+1}^{(1)} - (i_l + i_{l+1}). \tag{7.18}$$

The formula (7.17) and (7.18) show,

$$i_l + i_{l+1} \leq 0. \tag{7.19}$$

Since i_l is the right-most entry in d_{j+1}, there exist walls in position $l+1$. Therefore, by (7.19),

$$i_l + i_{l+1} < 0. \tag{7.20}$$

As in (ii), there are the following four cases (a)–(d) since $i_{k+1} = \pm t - 1$:

(a) $i_{k+1} = i_l = t-1$. (*i.e.* $i_k = -t$, $k+1$ and l are even.)

(b) $i_{k+1} = i_l = -t-1$. (*i.e.* $i_k = t$, $k+1$ and l are odd.)

(c) $i_{k+1} = t-1$ and $i_l = -t+1$. (*i.e.* $i_k = -t$, $k+1$ is even and l is odd.)

(d) $i_{k+1} = -t-1$ and $i_l = t+1$. (*i.e.* $i_k = t$, $k+1$ is odd and l is even)

The condition (a) or (b) is equivalent to that $l(d_{j+1})$ is odd and the condition (c) or (d) is equivalent to that $l(d_{j+1})$ is even. Thus, it is enough to show that if (a) or (b), d_{j+1} is critical and if (c) or (d), d_{j+1} is regular.
Let d_q be the domain including i_{l+1}. Applying (7.20) to these cases, we get

(a) $t(d_q) = -i_{l+1} > i_l = t-1 = t(d_{l+1})$. This implies that $t(d_{j+2}) = t$ and then d_{j+1} is a critical domain.

(b) $t(d_q) = i_{l+1} < -i_l = t+1 = t(d_{l+1})$. This implies that $t(d_{j+2}) = t$ and then d_{j+1} is a critical domain.

(c) $t(d_q) = i_{l+1} < -i_l = t-1 = t(d_{l+1})$. This implies that $t(d_{j+2}) = t-2$ and then d_{j+1} is a regular domain.

(d) $t(d_q) = -i_{l+1} > i_l = t+1 = t(d_{l+1})$. This implies that $t(d_{j+2}) = t+2$ and then d_{j+1} is a regular domain.

Since the cases (a)–(d) cover all possibilities for d_{j+1}, we obtain the desired results. □

Now, we set that for domains $d = i_k, \cdots i_l$ $(k \leq l)$ in a path p and $d' = j_s, \cdots, j_t$ $(s \leq t)$ in a path p', $d \subset d'$ if $s \leq k \leq l \leq t$ and $i_r = j_r$ for $r = k, \cdots, l$. We set $d = d'$ if and only if $d \subset d'$ and $d' \subset d$.

Lemma 7.14 *Suppose that for $p = (\cdots, i_{k-1}, i_k, i_{k+1}, \cdots) \in \mathcal{P}_m(n)$ $\tilde{f}_i p = (\cdots, i_{k-1}, \tilde{f}_i(i_k), i_{k+1}, \cdots)$ (resp. $\tilde{e}_i p = (\cdots, i_{k-1}, \tilde{e}_i(i_k), i_{k+1}, \cdots)$) and let $d_1, \cdots, d_{n-1}$ and $d'_1, \cdots, d'_{n-1}$ be the finite domains in p and $\tilde{f}_i p$ (resp. $\tilde{e}_i p$) respectively. In particular, let d_j be the domain including i_k. Then, we get*

(*i*) *If* $i \neq j, j+1$ (*resp.* $i \neq j-1, j$), *then* $d_i = d'_i$.

(*ii*) *If the domain* d_j *is finite, as a set* $d'_j \subset d_j$ *and* $d_j \setminus d'_j = \{i_k\}$ *and then*

$$l(d'_j) = l(d_j) - 1$$

(*iii*) *If the domain* d_{j+1} (*resp.* d_{j-1}) *is finite, as a set* $d_{j+1} \subset d'_{j+1}$ (*resp.* $d_{j-1} \subset d'_{j-1}$) *and* $d'_{j+1} \setminus d_{j+1} = \{\tilde{f}_i(i_k)\}$ (*resp.* $d'_{j-1} \setminus d_{j-1} = \{\tilde{e}_i(i_k)\}$) *and then*

$$l(d'_{j+1}) = l(d_{j+1}) + 1 \quad (\textit{resp. } l(d'_{j-1}) = l(d_{j-1}) + 1).$$

Proof. We shall see only the $\tilde{f}_1$ case since other cases can be shown similarly. By the proof of Lemma 7.13, we know that $i_k + i_{k+1} < 0$. We can also get $i_{k-1} + i_k \geq 0$. By the fact that $\tilde{f}_1(i_k) = i_k + 1$, we have

$$|\tilde{f}_1(i_k) + i_{k+1}| = |i_k + i_{k+1}| - 1 \quad \text{and} \quad |i_{k-1} + \tilde{f}_1(i_k)| = |i_{k-1} + i_k| + 1. \tag{7.21}$$

This means that one wall in position $k+1$ shifts to position k and the entry in the position k is transferred from d_j to d'_{j+1} by the action of $\tilde{f}_1$. The shifted wall is the $j+1$ th wall since it is on the right boundary of the domain d_j. Here note that a domain d_k is surrounded by k-th wall and $k+1$-th wall. Thus we obtain the desired results. □

Proof of Proposition 7.12.
For $p \in \mathcal{P}_m(n; \vec{t}; \vec{c})$, suppose that $\tilde{f}_i p = (\cdots, i_{k-1}, \tilde{f}_i(i_k), i_{k+1}, \cdots) \neq 0$. Let $d_0, d_1, \cdots, d_{n-1}, d_n$ and $d'_0, d'_1, \cdots, d'_{n-1}, d'_n$ be domains of p and $\tilde{f}_1 p$ respectively, in particular d_j be the domain including i_k (d_0, d_n, d'_0 and d'_n are infinite domains.). First let us show

$$t(d'_i) = t(d_i) \quad \text{for any } \ i = 1, 2, \cdots, n-1. \tag{7.22}$$

By Lemma 7.14 (i), we know that for $i \neq j, j+1$ such that $d_i = d'_i$ is non-zero length domain, $t(d'_i) = t(d_i)$. We shall consider the type of d'_{j+1}. If d_{j+1} and d'_{j+1} are infinite domains, $j+1 = 0$ or n then there is nothing to prove. Then we may assume that d_{j+1} and d'_{j+1} are finite domains. If $l(d_{j+1}) \geq 1$, there exists $a \in \mathbf{Z}$ such that i_a is included in both d_{j+1} and d'_{j+1} by Lemma 7.14 (iii). Then, in this case we get $t(d'_{j+1}) = t(d_{j+1})$. In the case $l(d_{j+1}) = 0$ if we assume that $t(d_j) = t$ and $t(d_{j+1}) = t+1$, by the proof of Lemma 7.13 related to (a) and (c), we get that $i_k = t$ and k is even. Then $\tilde{f}_1(i_k) = t+1$. This entry is included in d'_{j+1} and then $t(d'_{j+1}) = (-1)^k \tilde{f}_1(i_k) = (-1)^k(t+1) = t+1$. We can also easily see the case $t(d_{j+1}) = t-1$. Thus we get

$$t(d'_{j+1}) = t(d_{j+1}).$$

We shall consider the type of d'_j. As same as above, we may assume that d_j and d'_j are finite domains. If $l(d_j) \geq 2$, there exists $b \in \mathbf{Z}$ such that i_b is included

in both d_j and d'_j by Lemma 7.14 (ii). Then, in this case we get $t(d'_j) = t(d_j)$. If $l(d_j) = 1$, by Lemma 7.13 (ii) and Lemma 7.14 (ii), we get that d_j is a regular domain and $l(d'_j) = 0$. Since by the previous arguments we have already obtained $t(d'_i) = t(d_i)$ for $i \neq j$ such that d_i or d'_i is a non-zero length domain and d'_j is a regular domain by the remark under Definition 7.8, we get

$$t(d'_j) = t(d_j).$$

Thus we get $t(d_i) = t(d'_i)$ for all other zero-length domains. Then we obtain (7.22).

Next, let us show

$$c(d'_i) = c(d_i) \quad \text{for any} \quad i = 1, 2, \cdots, n-1. \tag{7.23}$$

By (7.22), d_i is a regular (resp. critical) domain if and only if d'_i is a regular (resp. critical) domain. Therefore, by Lemma 7.14 (i) we have

$$c(d'_i) = c(d_i) \quad \text{for} \quad i \neq j, j+1. \tag{7.24}$$

We shall consider the domain parameter $c(d'_j)$. We may assume that d_j and d'_j are finite domains as in the previous arguments. If d_j is a regular, d'_j is also regular and by Lemma 7.13 (ii) and Lemma 7.14 (ii) we have

$$l(d_j) = 2c_j + 1 \quad \text{and} \quad l(d'_j) = 2c_j. \tag{7.25}$$

Since d_j and d'_j are regular domains, the formula (7.25) implies

$$c(d'_j) = c_j = c(d_j). \tag{7.26}$$

If d_j is a critical domain, d'_j is also critical and by Lemma 7.13 (ii) and Lemma 7.14 (ii) we have

$$l(d_j) = 2c_j + 2 \quad \text{and} \quad l(d'_j) = 2c_j + 1. \tag{7.27}$$

Since d_j and d'_j are critical domains, the formula (7.27) implies

$$c(d'_j) = c_j = c(d_j). \tag{7.28}$$

As for d'_{j+1}, by using Lemma 7.13 (iii) and Lemma 7.14 (iii) we can also easily obtain

$$c(d'_{j+1}) = c_{j+1} = c(d_{j+1}). \tag{7.29}$$

Thus by (7.24), (7.26) (7.28) and (7.29) we get (7.23). Now, we have completed the proof of Proposition 7.12. $\square$

7.3 Extremal vectors in $\mathcal{P}_m(n;\vec{t};\vec{c})$

In this subsection, we shall describe all extremal vectors in $\mathcal{P}_m(n;\vec{t};\vec{c})$ explicitly.

Lemma 7.15 *Let B_1 and B_2 be normal crystals and $\phi : B_1 \to B_2$ be a strict morphism of crystal and we assume that $\phi(b) \neq 0$ for $b \neq 0$. We have that b is an extremal vector in B_1 if and only if $\phi(b)$ is an extremal vector in B_2.*

Proof. We assume that b is not an extremal vector in B_1 and $\phi(b)$ is an extremal vector in B_2. Then there exist $i, i_1, \cdots, i_k \in I$ such that

$$\tilde{e}_i S_{i_1} \cdots S_{i_k} b \neq 0 \quad \text{and} \quad \tilde{f}_i S_{i_1} \cdots S_{i_k} b \neq 0.$$

By the assumption that $\phi(b) \neq 0$ for $b \neq 0$, we get $\phi(\tilde{e}_i S_{i_1} \cdots S_{i_k} b) \neq 0$ and $\phi(\tilde{f}_i S_{i_1} \cdots S_{i_k} b) \neq 0$. Since ϕ is a morphism of crystal, we have

$$\tilde{e}_i S_{i_1} \cdots S_{i_k} \phi(b) \neq 0 \quad \text{and} \quad \tilde{f}_i S_{i_1} \cdots S_{i_k} \phi(b) \neq 0.$$

This contradicts the fact that $\phi(b)$ is an extremal vector. If b is an extremal vector in B_1 and $\phi(b)$ is not an extremal vector in B_2, by arguing similarly we can obtain a contradiction. $\square$

Let $\vec{t} = (t_1, \cdots, t_{n-1})$ be a m-domain configuration($m \in \mathbf{Z}$) and $\vec{c} = (c_1, \cdots, c_{n-1})$ be a sequence of non-negative integers. For $p \in \mathcal{P}_m(n;\vec{t};\vec{c}) \neq \emptyset$, let $d_1, \cdots, d_{n-1}$ be its finite domains. For a domain d_i we set

$$l_{\min}(d_j) := \begin{cases} 2c_i & \text{if } d_i \text{ is regular} \\ 2c_i + 1 & \text{if } d_i \text{ is critical} \end{cases} \tag{7.30}$$

Lemma 7.16 *For $p \in \mathcal{P}_m(n)$ let $\iota_1(p), \cdots, \iota_n(p)$ be its types of walls and set*

$$\begin{aligned} A_m(n;\vec{t};\vec{c}) &:= \left\{ p \in \mathcal{P}_m(n;\vec{t};\vec{c}) | \iota_1(p) = \cdots = \iota_n(p) \right\}. & (7.31) \\ E_m(n;\vec{t};\vec{c}) &:= \left\{ p \in \mathcal{P}_m(n;\vec{t};\vec{c}) | l(d_i) = l_{\min}(d_i) \ \ \textit{for any } i. \right\}. & (7.32) \end{aligned}$$

Then we get

$$A_m(n;\vec{t};\vec{c}) = E_m(n;\vec{t};\vec{c}). \tag{7.33}$$

Proof. For $p \in E_m(n;\vec{t};\vec{c})$ suppose that a domain d_j in p is a regular domain with non-zero length and set $t(d_j) = t$. Let i_a and i_b be the left-most entry and the right-most entry in d_j respectively. By (7.30), $l(d_j) = b - a + 1 = 2c_j > 0$. Thus, if a is even (resp. odd), b is odd (resp. even). Now we assume that a is even and b is odd. Then we have

$$t(d_j) = i_a = -i_b. \tag{7.34}$$

Let d_r and d_s be the domains including i_{a-1} and i_{b+1} respectively. We have

$$t(d_r) = -i_{a-1} \quad \text{and} \quad t(d_s) = i_{b+1}, \tag{7.35}$$

since $a-1$ is odd and $b+1$ is even. Because d_j is regular,

$$t(d_r) < t(d_j) < t(d_s) \text{ or } t(d_r) > t(d_j) > t(d_s). \tag{7.36}$$

Applying (7.34) and (7.35) to (7.36) we obtain

$$i_{a-1} + i_a > 0,\ i_b + i_{b+1} > 0 \text{ or } i_{a-1} + i_a < 0,\ i_b + i_{b+1} < 0.$$

This means that all walls in a and in $b+1$ have the same type. We can get the same result for the case that a is odd and b is even, and the case that d_j is critical. Repeating this for all domains with non-zero length, we know that all walls have same type in p. Thus, we have

$$E_m(n;\vec{t};\vec{c}) \subset A_m(n;\vec{t};\vec{c}). \tag{7.37}$$

Let p be an element of $A_m(n;\vec{t};\vec{c})$ and all walls in p be $+$. For a regular domain with non-zero length d_j in p let i_a and i_b be left-most entry and right-most entry in d_j respectively. Then we get

$$i_{a-1} + i_a > 0, \text{ and } i_b + i_{b+1} > 0. \tag{7.38}$$

Let d_r and d_s be as above. Since d_j is regular, we have (7.36). If a is even, $t(d_j) = i_a$ and $t(d_r) = -i_{a-1}$. By (7.38), we get $t(d_r) < t(d_j)$. Thus, by the assumption that d_j is regular, we have

$$t(d_r) < t(d_j) < t(d_s). \tag{7.39}$$

Furthermore, if b is even, $t(d_j) = i_b$ and $t(d_s) = -i_{b+1}$. Then this and (7.39) imply that $i_b + i_{b+1} < 0$. But this contradicts (7.38). Then b is odd and then $l(d_j) = b - a + 1$ is even. Since d_j is regular, this means

$$l(d_j) = 2c_j = l_{\min}(d_j).$$

By arguing similarly for other non-zero length domains, we obtain $l(d_i) = l_{\min}(d_i)$ for any i. Therefore, we get

$$A_m(n;\vec{t};\vec{c}) \subset E_m(n;\vec{t};\vec{c}). \tag{7.40}$$

By (7.37) and (7.40), we get the desired result. □

Proposition 7.17 *Let E be the set of all extremal vectors in $\mathcal{P}_m(n;\vec{t};\vec{c})$. Then we have*

$$E = A_m(n;\vec{t};\vec{c}) = E_m(n;\vec{t};\vec{c}). \tag{7.41}$$

Proof. By the definition of the map ψ given in (6.1), we know that $\psi(p) \neq 0$ for $p \in \mathcal{P}_m(n)$. Therefore, by Proposition 4.3, Theorem 6.1 and Lemma 7.15, we get

$$A_m(n;\vec{t};\vec{c}) = E. \tag{7.42}$$

By Lemma 7.16, we obtain the desired result. $\square$

Let $p_l^{(\pm)}$ be paths given in the proof of Proposition 7.11 and set

$$E' := \{p_l^{(\pm)}\}_{l\in\mathbf{Z}}.$$

Theorem 7.18 *We have*

$$E = E' = A_m(n;\vec{t};\vec{c}) = E_m(n;\vec{t};\vec{c}). \tag{7.43}$$

Proof. By the definiton of $p_l^{(\pm)}$ in the proof of Proposition 7.11 and Proposition 7.17, we get $E' \subset E$ easily. For $p_l^{(\epsilon)}$ ($\epsilon = \pm$, $l \in \mathbf{Z}$) let $k_1^{\epsilon,l}, \cdots, k_n^{\epsilon,l}$ be the positions of walls in $p_l^{(\epsilon)}$. By (7.5), (7.6) and the way of ordering, we get

$$\begin{array}{rcl} (k_1^{+,l}, \cdots, k_n^{+,l}) & = & (k_1^{-,l}+1, \cdots, k_n^{-,l}+1), \\ (k_1^{-,l}, \cdots, k_n^{-,l}) & = & (k_1^{+,l-1}+1, \cdots, k_n^{+,l-1}+1). \end{array} \tag{7.44}$$

Let p be an element in $E_m(n;\vec{t};\vec{c})$ and $(k_1, \cdots, k_n)$ be the positions of walls in p. By the definiton of $E_m(n;\vec{t};\vec{c})$, we know that for any ϵ and l, $k_{j+1}^{\epsilon,l} - k_j^{\epsilon,l} = l_{\min}(d_j) = (2c_j \text{ or } 2c_j+1) = k_{j+1} - k_j$. Therefore, by (7.44), there exist $\epsilon \in \{\pm\}$ and $l \in \mathbf{Z}$ such that $(k_1, \cdots, k_n) = (k_1^{\epsilon,l}, \cdots, k_n^{\epsilon,l})$. Now, since the domain types are fixed, the entries in p are automatically determined and it coincides with the ones in $p_l^{(\epsilon)}$. This means that $p = p_l^{(\epsilon)}$ and then

$$E_m(n;\vec{t};\vec{c}) \subset E'.$$

Now, we have completed the proof. $\square$

Remark. By Lemma 7.14, we know that a $(-)$(resp. $(+)$) wall in a path is shifted by one to the left direction by the action of $\tilde{f}_1$ (resp. $\tilde{f}_0$) and by the definition of $p_l^{(\pm)}$ $(k_1^+, \cdots, k_n^+) = (k_1^- + 1, \cdots, k_n^- + 1)$, where $(k_1^\pm, \cdots, k_n^\pm)$ are sequences of the positions of walls in $p_l^{(\pm)}$. Therefore, we have

$$\tilde{f}_1^n p_l^{(-)} = p_{l-1}^{(+)} \quad \text{and} \quad \tilde{f}_0^n p_l^{(+)} = p_l^{(-)}. \tag{7.45}$$

Thus, we have

$$S_1 p_l^{(-)} = \tilde{f}_1^n p_l^{(-)} = p_{l-1}^{(+)} \quad \text{and} \quad S_0 p_l^{(+)} = \tilde{f}_0^n p_l^{(+)} = p_l^{(-)}. \tag{7.46}$$

By these (7.45) and (7.46), we get

$$S_1 p_l^{(-)} = p_{l-1}^{(+)}, \; S_0 p_l^{(+)} = p_l^{(-)}, \; S_1 p_l^{(+)} = p_{l+1}^{(-)} \text{ and } \; S_0 p_l^{(-)} = p_l^{(+)}. \tag{7.47}$$

Thus, we obtain the following result.

Corollary 7.19 *$\mathcal{P}_m(n;\vec{t};\vec{c})$ is a connected component in $\mathcal{P}_m$.*

Proof. By the remark as above, we know that E' is connected and then any extremal vector in $\mathcal{P}_m(n;\vec{t};\vec{c})$ is connected to each other. Therefore, by Theorem 3.7 and Proposition 7.12, we know that $\mathcal{P}_m(n;\vec{t};\vec{c})$ is connected. $\square$

Example 7.20

$$
\begin{array}{cccccccccccc}
 & \cdots & 0 & 0 & 0\overset{+}{|} & 1 & -1\overset{+}{|} & 2 & -2\overset{+}{|} & 3 & -3 & \cdots \\
\overset{S_0}{\longleftrightarrow} & \cdots & 0 & 0\overset{-}{|} & -1 & 1\overset{-}{|} & -2 & 2\overset{-}{|} & -3 & 3 & -3 & \cdots \\
\overset{S_1}{\longleftrightarrow} & \cdots & 0\overset{+}{|} & 1 & -1\overset{+}{|} & 2 & -2\overset{+}{|} & 3 & -3 & 3 & -3 & \cdots
\end{array}
$$

7.4 Affinization of the Path-Spin Correspondence

In Sec.6 we introduced the path-spin correspondence. In this subsection, we shall affinize it, that is, the path-spin correspondence in Sec.6, which is a morhpism of classical crystal, is lifted to a morphism of affine crystals.

Let $B = \{+,-\}$ be the classical crystal as in Example 4.2.

Lemma 7.21 (see 2.2) *The set of all extremal vectors in* $\mathrm{Aff}(B^{\otimes n})$ *is given by*

$$\{z^k \otimes (+)^{\otimes n}, z^k \otimes (-)^{\otimes n}\}_{k\in\mathbf{Z}}. \tag{7.48}$$

Proof. By (4.2), (4.3) and (4.6), we have

$$
\begin{aligned}
S_1(z^k \otimes (\pm)^{\otimes n}) &= z^k \otimes (\mp)^{\otimes n}, \\
S_0(z^k \otimes (\pm)^{\otimes n}) &= z^{k\pm n} \otimes (\mp)^{\otimes n}.
\end{aligned}
$$

By (4.3) and (4.5), we get for any k

$$\tilde{e}_1(z^k\otimes(+)^{\otimes n}) = \tilde{f}_0(z^k\otimes(+)^{\otimes n}) = \tilde{e}_0(z^k\otimes(-)^{\otimes n}) = \tilde{f}_1(z^k\otimes(-)^{\otimes n}) = 0. \tag{7.49}$$

Thus, we get the desired result. $\square$

Now we shall consider the affinization of the morphism ψ. For a level 0 affine weight $\lambda = m(\Lambda_0 - \Lambda_1) + l\delta \in P$ $(l, m \in \mathbf{Z})$ by Remark (ii) in 4.2, we have that Ua_λ has a U'-module structure and its crystal $B(Ua_\lambda)$ is described by $\mathcal{P}_m$ as a classical crystal (that is, $B(Ua_\lambda) \cong B(U'a_{cl(\lambda)})$ as a classical crystal). Originally, the crystal $B(Ua_\lambda)$ holds an affine crystal structure. We shall recover its affine crystal structure in terms of path. For this purpose we shall introduce the *energy function* (See [3], [4], [11]).

Definition 7.22 ([11]) *Let B be a classical crystal. A $\mathbf{Z}$-valued function H on $B \otimes B$ is called an energy function on B if for any $i \in I$ and $b \otimes b' \in B \otimes B$ such that $\tilde{e}_i(b \otimes b') \neq 0$, we have*

$$
\begin{aligned}
H(\tilde{e}_i(b \otimes b')) &= H(b \otimes b') \text{ if } i = 1, \\
&= H(b \otimes b') + 1 \text{ if } i = 0 \text{ and } \varphi_0(b) \geq \varepsilon_0(b'), \\
&= H(b \otimes b') - 1 \text{ if } i = 0 \text{ and } \varphi_0(b) < \varepsilon_0(b').
\end{aligned}
$$

For the case of $B = B_\infty$, by [11] Theorem 5.1, we can describe the function H explicitly as follows.

Proposition 7.23 *We set*

$$H((m) \otimes (n)) := \max\{m, -n\}.$$

This H is an energy function on B_∞.

By Theorem 4.9 in [11], we get the following theorem.

Theorem 7.24 *Let $(g_i)_{i\in\mathbf{Z}}$ be a m-ground-state path. For a level 0 affine weight $\lambda = m(\Lambda_0 - \Lambda_1) + l\delta \in P$ and $b \in B(Ua_\lambda)$ which corresponds to the m-path $p = (i_k)_{k\in\mathbf{Z}} \in \mathcal{P}_m$ as a classical crystal, we have the following formula,*

$$\begin{aligned} wt(b) = wt(p) = (\sum_{k\in\mathbf{Z}} i_{k-1} + i_k)(\Lambda_0 - \Lambda_1) \\ +(l + \textstyle\sum_{k\in\mathbf{Z}} k(\max\{i_{k-1}, -i_k\} - \max\{g_{k-1}, -g_k\}))\delta. \end{aligned} \tag{7.50}$$

Proof. By using the formula in [11, Theorem 4.9.] and the same type of the formula for $B(-\infty)$, we can easily derive

$$\begin{aligned} wt(b) = wt(p) = m(\Lambda_0 - \Lambda_1) + \sum_{k\in\mathbf{Z}} (af(wt(i_k)) - af(wt(g_k))) \\ +\{l + \textstyle\sum_{k\in\mathbf{Z}} k(H(i_{k-1} \otimes i_k) - H(g_{k-1} \otimes g_k))\}\delta. \end{aligned} \tag{7.51}$$

Since by the definition of path, the summations in (7.51) are finite, we get

$$\begin{aligned} wt(p) = m(\Lambda_0 - \Lambda_1) + \frac{1}{2} \sum_{i\in\mathbf{Z}} (af(wt(i_{k-1})) + af(wt(i_k)) \\ -af(wt(g_{k-1})) - af(wt(g_k))) + \{l + \textstyle\sum_{k\in\mathbf{Z}} k(H(i_{k-1} \otimes i_k) - H(g_{k-1} \otimes g_k))\}\delta. \end{aligned} \tag{7.52}$$

By applying Proposition 7.23 and

$$\begin{aligned} af(wt(i_{k-1})) + af(wt(i_k)) &= 2(\Lambda_0 - \Lambda_1)(i_{k-1} + i_k), \\ af(wt(g_{k-1})) + af(wt(g_k)) &= \begin{cases} 2m(\Lambda_0 - \Lambda_1) & k = 0, \\ 0 & k \neq 0, \end{cases} \end{aligned}$$

to (7.52), we obtain the desired result. $\square$

For a level 0 weight $\lambda = m(\Lambda_0 - \Lambda_1) + l\delta$, we denote $\mathcal{P}_{m,l}$ for a set of path corresponding to an element of $B(Ua_\lambda)$, *i.e.* as a set $\mathcal{P}_{m,l}$ is equal to $\mathcal{P}_m$ and a weight is given by (7.50).

By using this formula, we get $\widehat{\psi}$: the affinization of ψ as follows: For $p \in \mathcal{P}_{m,l}$ a map $\widehat{\psi}$ is given by

$$\begin{array}{rcl} \widehat{\psi} : \mathcal{P}_{m,l} & \longrightarrow & \mathrm{Aff}(B^{\otimes n}) \\ p & \mapsto & z^{\langle d, wt(p)\rangle} \otimes \psi(p), \end{array} \tag{7.53}$$

Let us denote also $\tilde{\psi}$ for the restriction of $\widehat{\psi}$ to $\mathcal{P}_{m,l}(n;\vec{t};\vec{c})$, where $\mathcal{P}_{m,l}(n;\vec{t};\vec{c})$ is equal to $\mathcal{P}_m(n;\vec{t};\vec{c})$ as a set and a weight is given by (7.50).

Theorem 7.25 *(i) The map $\widehat{\psi}$ and $\tilde{\psi}$ are strict morphisms of affine crystals.*

(ii) The map $\tilde{\psi}$ is an injective morphism of affine crystal.

Proof. (i) It is trivial by (4.2), Theorem 6.1 and Proposition 7.12.

(ii) In order to show (ii) we shall see the following lemmas :

Lemma 7.26 *Let E be the set of all extremal vectors in $\mathcal{P}_{m,l}(n;\vec{t};\vec{c})$. If the map $\tilde{\psi}|_E$ is injective, the map $\tilde{\psi}$ is injective.*

Proof. We assume that $\tilde{\psi}$ is not injective. Then there exist $p_1, p_2 \in \mathcal{P}_{m,l}$ such that $p_1 \neq p_2$ and $\tilde{\psi}(p_1) = \tilde{\psi}(p_2)$. We set $b^* := \tilde{\psi}(p_1) = \tilde{\psi}(p_2) \in \mathrm{Aff}(B^{\otimes n})$. Due to the connectedness of $B^{\otimes n}$, for this b^* there exist $\tilde{x}_{i_1}, \cdots, \tilde{x}_{i_l} \in \{\tilde{e}_i, \tilde{f}_i\}_{i=0,1}$ and an extremal vector $v \in \mathrm{Aff}(B^{\otimes n})$ such that

$$v = \tilde{x}_{i_1} \cdots \tilde{x}_{i_l}(b^*). \tag{7.54}$$

Since $v \neq 0$ is an extremal vector, by Theorem 7.25 (i) and Lemma 7.15, we have that both $\tilde{x}_{i_1} \cdots \tilde{x}_{i_l} p_1 \neq 0$ and $\tilde{x}_{i_1} \cdots \tilde{x}_{i_l} p_2 \neq 0$ are elements in E. The injectivity of $\tilde{\psi}|_E$ means

$$\tilde{\psi}(\tilde{x}_{i_1} \cdots \tilde{x}_{i_l} p_1) \neq \tilde{\psi}(\tilde{x}_{i_1} \cdots \tilde{x}_{i_l} p_2).$$

since $p_1 \neq p_2$ and then $\tilde{x}_{i_1} \cdots \tilde{x}_{i_l} p_1 \neq \tilde{x}_{i_1} \cdots \tilde{x}_{i_l} p_2$. But this contradicts the fact that

$$\tilde{x}_{i_1} \cdots \tilde{x}_{i_l} \tilde{\psi}(p_1) = v = \tilde{x}_{i_1} \cdots \tilde{x}_{i_l} \tilde{\psi}(p_2).$$

We have completed the proof of Lemma 7.26. □

Proof of Theorem 7.25 (ii) For a path $p \in \mathcal{P}_m$ let $\iota_1(p), \cdots, \iota_n(p)$ be a sequence of the types of the walls in p. We set

$$E_+ \;:=\; \{p \in E = E_m(n;\vec{t};\vec{c}) | \iota_i(p) = +,\ i = 1, \cdots, n\}, \tag{7.55}$$

$$E_- \;:=\; \{p \in E = E_m(n;\vec{t};\vec{c}) | \iota_i(p) = -,\ i = 1, \cdots, n\}. \tag{7.56}$$

These $E_\pm$ coincides with $\{p_l^{(\pm)}\}_{l \in \mathbf{Z}}$ respectively. By (7.46) and (7.47), we have the following;

Lemma 7.27 *For any $p_k^{(\epsilon_1)} \neq p_l^{(\epsilon_2)}$ ($\epsilon_1, \epsilon_2 = \pm$ and $k, l \in \mathbf{Z}$) we have*

$$wt(p_k^{(\epsilon_1)}) \neq wt(p_l^{(\epsilon_2)}). \tag{7.57}$$

Proof. If $\epsilon_1 \neq \epsilon_2$, $wt(p_k^{(\epsilon_1)}) \neq wt(p_l^{(\epsilon_2)})$ since $wt(p_k^{(-)}) = n(\Lambda_0 - \Lambda_1) + D_1\delta$ and $wt(p_l^{(+)}) = n(\Lambda_1 - \Lambda_0) + D_2\delta$ where D_1 and D_2 are some integers. Then we may assume that $\epsilon_1 = \epsilon_2$. We set $\epsilon_1 = \epsilon_2 = +$ and $k < l$. By (7.47), we have $S_0 S_1 p_l^{(+)} = p_{l-1}^{(+)}$. This means

$$(S_1 S_0)^{l-k} p_l^{(+)} = p_k^{(+)}.$$

Since $S_1 S_0 = \tilde{f}_1^n \tilde{f}_0^n$ for $p_l^{(+)}$, we get

$$\langle d, wt(p_l^{(+)})\rangle - \langle d, wt(p_k^{(+)})\rangle = (l-k)n > 0. \tag{7.58}$$

Now, we have completed the proof of Lemma 7.27. □

This lemma implies that any extremal vector in E has different weight each other. Since the morphism of affine crystal $\tilde{\psi}$ preserves weight, now we obtain the injectivity of the map $\tilde{\psi}|_E$. Therefore, by Lemma 7.26, we get the injectivity of $\tilde{\psi}$. We have completed the proof of Theorem 7.25. □

By the formula $S_1 p_l^{(-)} = p_{l-1}^{(+)}$ and $S_1 p_l^{(+)} = p_{l+1}^{(-)}$ in (7.47), we get

$$\langle d, p_l^{(-)}\rangle = \langle d, p_{l-1}^{(+)}\rangle.$$

By this and (7.58), for any extremal vectors $p_1, p_2 \in \mathcal{P}_{m,l}(n; \vec{t}; \vec{c})$, we have

$$\langle d, wt(p_1)\rangle \equiv \langle d, wt(p_2)\rangle \pmod{n}.$$

By this formula, we obtain the following;

Corollary 7.28 (*i*) *Set* $I_n := \{0, 1, \cdots, n-1\}$ *and let* $E_{m,l}(n; \vec{t}; \vec{c})$ *be the set of all extremal vectors in* $\mathcal{P}_{m,l}(n; \vec{t}; \vec{c})$. *Then there exists unique* $i \in I_n$ *such that*

$$\tilde{\psi}(E_{m,l}(n; \vec{t}; \vec{c})) = \{z^{i+kn} \otimes (\pm)^{\otimes n}\}_{k \in \mathbf{Z}}.$$

(*ii*) *Let us denote* $\mathrm{Aff}(B^{\otimes n})_i$ *for a connected component of* $\mathrm{Aff}(B^{\otimes n})$ *generated by extremal vectors* $\{z^{i+kn} \otimes (\pm)^{\otimes n}\}_{k \in \mathbf{Z}}$. *Then as a morphism of affine crystals,*

$$\tilde{\psi} : \mathcal{P}_{m,l}(n; \vec{t}; \vec{c}) \xrightarrow{\sim} \mathrm{Aff}(B^{\otimes n})_i.$$

Now, we shall summarize the classification of paths in $\mathcal{P}_{m,l} \cong B(U_q(\mathfrak{g})a_\lambda)$ $(\lambda = m(\Lambda_0 - \Lambda_1) + l\delta)$. By Corollary 7.28 (ii), if we fix one connected component in $\mathcal{P}_{m,l}(n)$, each element in the component is classified by $\mathrm{Aff}(B^{\otimes n})_i$. Since $\mathrm{Aff}(B^{\otimes n})_i$ is generated by $\{z^{i+kn} \otimes (\pm)^{\otimes n}\}_{k \in \mathbf{Z}}$, any element in $\mathrm{Aff}(B^{\otimes n})_i$ is in the following form:

$$z^{i_{\iota_1, \cdots, \iota_{n-1}, \iota} + kn} \otimes (\iota_1) \otimes \cdots \otimes (\iota_{n-1}), \tag{7.59}$$

where k is an integer called *depth parameter* and $i_{\iota_1,\cdots,\iota_{n-1},l} \in I_n$ is determined only by $\iota_1, \cdots, \iota_{n-1}, l$ (if $(\iota_1, \cdots, \iota_{n-1}) = (\pm, \cdots, \pm)$, $i_{\iota_1,\cdots,\iota_{n-1},l} = i$.).

Therefore, for given $m, l \in \mathbf{Z}$, by the following parameters:

$$
\begin{aligned}
&n \in \mathbf{Z}_{\geq 0} \text{ with } n - |m| \in 2\mathbf{Z}_{\geq 0} \text{ (the total number of walls),} \\
&(t_1, \cdots, t_{n-1}) \text{ in } m\text{-domain configuration (domain types),} \\
&(c_1, \cdots, c_{n-1}) \in \mathbf{Z}^n_{\geq 0} \text{ (domain parameters),} \\
&(\iota_1, \cdots, \iota_{n-1}) \ (\iota_j = \pm) \text{ (types of walls),} \\
&k \in \mathbf{Z} \text{ (depth parameter),}
\end{aligned}
$$

every path in $\mathcal{P}_{m,l}$ is uniquely classified.

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